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A NOTE ON CLIFFORD'S DERIVATION OF BI-QUATERNIONS

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1. Introduction

In 1873, Clifford introduced the dual numbers [Clifford 1873]. Similarly to Hamilton, who defined the quaternion as the ratio of two vectors, Clifford searched for the ratio of two motors (these terms are redefined in Sec. 2). He adhered to Hamilton's path even though some steps of his derivation, as he himself pointed out, were not general but require a particular geometry.

Hamilton searched for the ratio of two vectors, namely, a vector, **v**, multiplied by a quaternion, $\boldsymbol{q}$, results in another vector, **v**':

$$\mathbf{v}' = \boldsymbol{q}\ \mathbf{v} \tag{1}$$

As can easily be seen, this equation is correct only when the vector is perpendicular to the quaternion's rotation axis , **n**, and when the quaternion is parametrized by the full (and not half) angle of rotation, θ, i. e.,

$$\boldsymbol{q} = [\cos\theta\ ,\ \mathbf{n}\ \sin\theta] \tag{2}$$

Applying the rules of quaternion multiplication to Eq. (1),

$$\mathbf{v}' = [q_0\ ,\ \mathbf{q}]\ [0\ ,\ \mathbf{v}] = [-\mathbf{q}\ \mathbf{v},\ q_0\ \mathbf{v} + \mathbf{q} \times \mathbf{v}] \tag{3}$$

and keeping in mind that a vector is denoted according to Hamilton as a pure quaternion (a quaternion with a scalar part equal to zero), then **v**' results in a pure quaternion only if $\boldsymbol{q}$ is perpendicular to **v**.

The same route was taken by Clifford. He looked for the ratio of two motors, namely, the operation that converts one motor into the other, and he used Hamilton's notation of the ratio of two vectors. Even though he knew that using the quaternion as a ratio of two vectors is correct only when the quaternion is perpendicular to the vectors - "...it is very important to remark that so long as AF means a *vector* not perpendicular to the axis of q, the expression q.AF has no meaning at all." (page 383) - nevertheless he continued with this derivation and eventually (and fortunately) arrived at the definition of the dual unit and bi-quaternions.

The purpose of this paper is to show that by using the formula of a general rotation it is possible to generalize Clifford's derivation for transformation of motors by a general bi-quaternion which, as we know, is pre-multiplication by a bi-quaternion and post-multiplication by its inverse.

This obviously has only historical meanings since by applying the principle of transference [Kotelnikov 1895, Study 1903, Dimentberg 1965, Martinez and Duffy 1993] to dualize real quaternion operations, bi-quaternions have been used in a correct manner, for quite a long time. It is interesting to note that by the time Clifford published his paper (1873), the physical meaning of the quaternion in context of finite rotation was already known some forty three years before [Cayley 1845, Cheng and Gupta 1989] and could be used by Clifford to generalize his result.

2. Restatement of the problem and Clifford's derivation of the dual number unit

Clifford's goal was to determine the operation that converts one motor into another which he concluded to be a bi-quaternion. To follow his derivation and the more general one given in Section 3, we rewrite, for convenience, some of his definitions.

A *vector* is a quantity with magnitude and direction (e.g. linear velocity or moment).

A *rotor* is a quantity with magnitude, direction, and position (e.g. rotational velocity about a fixed axis or force along line of action).

A *motor* is the sum of two or more rotors, which can be represented as a wrench or twist about a certain screw. For example, the sum of arbitrary system of forces is, in general, not a force but a combination of force and moment.

We start by repeating the basic arguments in Clifford's paper.

Analogous to Hamilton's "tensor-versor " operation, which converts one vector into the other, Clifford defined the "tensor-twist" for non-intersecting rotors. Since a rotor has a specific direction and position, a "tensor-twist" - a screw motion about the common perpendicular and stretching (or compressing) along the resulting rotor axis - is the operation which converts one rotor into the other.

Mathematically, this is given by:

$$T\,\mathbf{b} = \mathbf{a} \tag{4}$$

where $\mathbf{a}$ and $\mathbf{b}$ are rotors and T is the "tensor-twist" operation.
"Tensor - twist" is also the operation which relates two equal-pitched motors (a motor can be considered as a rotor associated with a pitch).
Since motors are the sum of two or more rotors, it can be written as:

$$\mathbf{A} = \mathrm{m}\,\mathbf{a} + \mathrm{n}\,\mathbf{b} \tag{5}$$

where m and n are scalars and $\mathbf{a}$ and $\mathbf{b}$ are rotors.
Assuming that the "tensor-twist" operator in a linear one, the following equation holds:

$$T(\mathrm{m}\,\mathbf{a} + \mathrm{n}\,\mathbf{b}) = \mathrm{m}\,\mathbf{c} + \mathrm{n}\,\mathbf{d} \tag{6}$$

where $T\,\mathbf{a} = \mathbf{c}$ and $T\,\mathbf{b} = \mathbf{d}$.

To arrive at his goal, namely, to find the ratio of two motors Clifford "divided" a motor **B** by a motor **A**.
Let **B**' be a motor with rotor part equal to motor **B** and with the same pitch as motor **A**, then:

$$\mathbf{B} = \mathbf{B}' + \beta \tag{7}$$

where β is a vector parallel to the axis of **B** (adding vector β to **B'** changes its pitch and maintain its direction and position).
Dividing by **A** while keeping in mind that **B'** and **A** have the same pitch one obtains:

$$\mathbf{B} / \mathbf{A} = T + \beta / \mathbf{A} \tag{8}$$

Observing the rightmost term, Clifford realized that one needs to apply a special unit which when applied to a motor converts it to a vector parallel and proportional to its rotor part. This unit is known as the dual number unit, ε, with the special property:

$$\varepsilon^2 = 0 , \tag{9}$$

reflecting the fact that when it is applied twice to a motor, the result is zero.

3. Transformation of a motor by a general bi-quaternion

At this point, we diverge from Clifford's derivation since he replaced β by $\boldsymbol{q}\ \varepsilon\,\mathbf{A}$ where $\varepsilon\,\mathbf{A}$ is a vector and $\boldsymbol{q}$ is a quaternion that changes $\varepsilon\,\mathbf{A}$ into β. As mentioned above and as Clifford himself noted, $\boldsymbol{q}\ \varepsilon\,\mathbf{A}$ is equal to β only when $\boldsymbol{q}$ is perpendicular to the plane through $\varepsilon\,\mathbf{A}$ and β.

We will next employ the expression of a general rotation:

$$\varepsilon\mathbf{A} = \boldsymbol{q}\ \beta\ \boldsymbol{q}^{-1} \tag{10}$$

Writing a motor in an equivalent way as a rotor through the origin plus a vector, it is possible to rotate both the rotor and the vector by quaternions. Hence, for the two motors: $\mathbf{a} + \varepsilon\,\mathbf{b}$; $\mathbf{c} + \varepsilon\,\mathbf{d}$, where **a** and **c** are rotors and $\varepsilon\,\mathbf{b}$ and $\varepsilon\,\mathbf{d}$ are vectors, there exists a quaternion $\boldsymbol{q}$ which satisfies:

$$\boldsymbol{c} = \boldsymbol{q}\ \boldsymbol{a}\ \boldsymbol{q}^{-1} \tag{11}$$

Note that $\boldsymbol{a}$ is treated here as a quaternion composed of a vector part equal to **a** and a zero scalar part. Then,

$$\boldsymbol{q}(\boldsymbol{a} + \varepsilon\,\boldsymbol{b})\boldsymbol{q}^{-1} = \boldsymbol{c} + \boldsymbol{q}\,\varepsilon\,\boldsymbol{b}\boldsymbol{q}^{-1} \tag{12}$$

Defining a new vector (as pure quaternion) $\boldsymbol{d} - \boldsymbol{q}\boldsymbol{b}\boldsymbol{q}^{-1}$ it can be related through another quaternion, $\boldsymbol{r}$, to **a** :

$$\boldsymbol{r}\ \boldsymbol{a}\ \boldsymbol{r}^{-1} = \boldsymbol{d} - \boldsymbol{q}\ \boldsymbol{b}\ \boldsymbol{q}^{-1} \tag{13}$$

Multiplying the above equation by ε and adding it to equation (12), the following is obtained:

$$\boldsymbol{q}\ (\boldsymbol{a}+\varepsilon\boldsymbol{b})\,\boldsymbol{q}^{-1}+\varepsilon\boldsymbol{r}\ \boldsymbol{a}\ \boldsymbol{r}^{-1}=\boldsymbol{c}+\varepsilon\boldsymbol{d} \tag{14}$$

which can be cast into

$$\boldsymbol{q}\ (\boldsymbol{a}+\varepsilon\boldsymbol{b})\,\boldsymbol{q}^{-1}+\varepsilon\boldsymbol{r}\ (\boldsymbol{a}+\varepsilon\boldsymbol{b})\,\boldsymbol{r}^{-1}=\boldsymbol{c}+\varepsilon\boldsymbol{d}\,. \tag{15}$$

Let's assume, for a moment, that the second (dual) term on the left hand side of this equation can be written as:

$$\boldsymbol{r}\ (\boldsymbol{a}+\varepsilon\boldsymbol{b})\,\boldsymbol{r}^{-1}=\boldsymbol{q}\ (\boldsymbol{a}+\varepsilon\boldsymbol{b})\ \boldsymbol{p}^{-1}+\boldsymbol{p}\ (\boldsymbol{a}+\varepsilon\boldsymbol{b})\,\boldsymbol{q}^{-1} \tag{16}$$

where $\boldsymbol{p}$ is another quaternion.
Then Eq. (15) becomes

$$(\boldsymbol{q}+\varepsilon\,\boldsymbol{p})\,(\boldsymbol{a}+\varepsilon\boldsymbol{b})\,(\boldsymbol{q}+\varepsilon\,\boldsymbol{p})^{-1}=\boldsymbol{c}+\varepsilon\,\boldsymbol{d} \tag{17}$$

which is exactly in the form we sought since it can be compactly written as

$$\hat{\boldsymbol{q}}\ \hat{\boldsymbol{a}}\,\hat{\boldsymbol{q}}^{-1}=\hat{\boldsymbol{c}} \tag{18}$$

with ^ denotes a dual quantity, and $\hat{\boldsymbol{a}}=(\boldsymbol{a}+\varepsilon\boldsymbol{b})$; $\hat{\boldsymbol{c}}=(\boldsymbol{c}+\varepsilon\boldsymbol{d})$.

This expression is a dualization of the known formula of a general rotation.

It reveals what we had known to be the correct answer; the operation that converts one motor into the other is a pre-multiplication by a bi-quaternion (not necessarily a perpendicular one) and post-multiplication by its inverse.

What remains to prove is that the expression $\boldsymbol{r}\ (\boldsymbol{a}+\varepsilon\boldsymbol{b})\,\boldsymbol{r}^{-1}$ can be written as $\boldsymbol{q}\ (\boldsymbol{a}+\varepsilon\boldsymbol{b})\ \boldsymbol{p}^{-1}+\boldsymbol{p}\ (\boldsymbol{a}+\varepsilon\boldsymbol{b})\,\boldsymbol{q}^{-1}$ (Eq.(16)).

Keeping in mind that a (dual) vector is given here as a pure quaternion which is actually a binary rotation (a rotation by 180 degrees), then if $\hat{\boldsymbol{a}}=(\boldsymbol{a}+\varepsilon\boldsymbol{b})$ is a binary rotation so should be also the right hand side of Eq. (16). If this is the case, then obviously the equation holds since any (dual) vector $\hat{\mathbf{a}}$ can be converted to another (dual) vector by some quaternion $\boldsymbol{r}$)the left hand side of Eq.(16)).

Hence, all that is needed in order to complete the proof is to show that the expression on the right hand side of Eq.(16) is a pure quaternion.
Expanding this expression and using quaternion multiplication rules, one obtains:

$$\begin{aligned}\boldsymbol{q}\hat{\boldsymbol{v}}\,\boldsymbol{p}^{-1}+\boldsymbol{p}\,\hat{\boldsymbol{v}}\,\boldsymbol{q}^{-1}=&[-\mathbf{p}\hat{\mathbf{v}}\mathrm{q}_0+(\mathrm{p}_0\hat{\mathbf{v}}+\mathbf{p}\mathrm{x}\hat{\mathbf{v}})\mathbf{q}\,,\,(\mathbf{p}\hat{\mathbf{v}})\mathbf{q}+\mathrm{q}_0(\mathrm{p}_0\hat{\mathbf{v}}+\mathbf{p}\mathrm{x}\hat{\mathbf{v}})+(\mathrm{p}_0\hat{\mathbf{v}}+\mathbf{p}\mathrm{x}\hat{\mathbf{v}})\mathrm{x}(-\mathbf{q})]+\\&[-\mathbf{q}\hat{\mathbf{v}}\mathrm{p}_0+(\mathrm{q}_0\hat{\mathbf{v}}+\mathbf{q}\mathrm{x}\hat{\mathbf{v}})\mathbf{p}\,,\,(\mathbf{q}\hat{\mathbf{v}})\mathbf{p}+\mathrm{p}_0(\mathrm{q}_0\hat{\mathbf{v}}+\mathbf{q}\mathrm{x}\hat{\mathbf{v}})+(\mathrm{q}_0\hat{\mathbf{v}}+\mathbf{q}\mathrm{x}\hat{\mathbf{v}})\mathrm{x}(-\mathbf{p})]\end{aligned} \tag{19}$$

where $\hat{\mathbf{v}}$ is the dual vector $\mathbf{a}+\varepsilon\,\mathbf{b}$, $\hat{\boldsymbol{v}}$ is its dual pure quaternion, and $\boldsymbol{q}=[\mathrm{q}_0,\mathbf{q}]$, $\hat{\boldsymbol{v}}=[0,\hat{\mathbf{v}}]$, and $\boldsymbol{p}=[\mathrm{p}_0,\mathbf{p}]$.

After some algebra, this equation indeed results in a pure quaternion (the exact value of the vector part is, actually, immaterial),

$$q\hat{v}\,p^{-1} + p\,\hat{v}\,q^{-1} = \\ [0\,,\,2((\mathbf{p}\hat{\mathbf{v}})\mathbf{q} + (\mathbf{q}\hat{\mathbf{v}})\mathbf{p} - (\mathbf{qp})\hat{\mathbf{v}} + q_0\,p_0\,\hat{\mathbf{v}} + q_0(\mathbf{p}\times\hat{\mathbf{v}}) + p_0\,(\mathbf{q}\times\hat{\mathbf{v}}))] \tag{20}$$

which completes the proof.

It should be noted that the quaternions used in this derivation are not necessarily unit quaternions, since they are used to convert one vector (rotor) into the other, and hence, they are need not be magnitude preserving entities.

4. Conclusions

The path Clifford took in his derivation of dual numbers, and their subsequent application to bi-quaternions, used Hamilton's notation of quaternions. Since the application of Hamilton's notation to rotation of vectors is correct only for vectors and quaternions that are mutually perpendicular, it follows that Clifford's proof has the same limitation for transformation of motors by bi-quaternions.

In this paper, Clifford's proof is extended to transformation of motors by a general bi-quaternion. The notation of a general rotation is used and it is shown that the operation that relates two motors is a pre-multiplication by a bi-quaternion and post-multiplication by its inverse. This was obtained by defining any motor as a rotor and a vector through the origin which allows the application of a quaternion as a rotating and stretching operator. It is then shown that pre-and post-multiplication of a motor (given as a pure dual quaternion) by a dual quaternion and its inverse respectively, transforms a pure dual quaternion into another pure dual quaternion for a general angle between the motor and the dual quaternion.

It should be noted that this formulation has been long used in its correct form through the application of the principle of transference to rotational operation. This paper shows that the same result could, however, be obtained through Clifford's derivation if quaternion were used in their rotational operation sense, rather then as a ratio of two vectors.

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