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Derivation of dual forces in robot manipulators

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Abstract

According to the principle of transference, a compact three-dimensional representation of a rigid body kinematics is obtained by substituting dual for real numbers. This representation has recently been applied to robotics where in addition to its compactness, it allows to constitute the Jacobian matrix explicitly from the product of the dual transformation matrices with no additional computation.

This paper introduces the generalized Jacobian matrix. This matrix consists of the complete dual transformation matrices as opposed to the regular Jacobian matrix which consists of specific columns only. The generalized Jacobian matrix relates force and moment at the end-effector to force and moment in all directions, at the joints. It is therefore possible to use the dual transformation matrices to derive, with no additional computation, the full force and moment vector at the robot's joints. Furthermore, the generalized Jacobian matrix also relates motion in all directions at the joints to the motion of the end-effector, an essential relation required at the design stage of robot manipulators (in particular, flexible ones). An extension of these kinematics and statics schemes into dynamics is possible by applying the dual inertia operator as is shown by an example of a three degrees-of-freedom robot.

1. Introduction

Robot manipulators are actuated by motors which are located at, or their motion is transmitted to, the robot's joint axes. For control purposes, it is necessary to calculate the loads developed at the joint axes in order to select the appropriate motors and to control the robot's motion. Utilizing the principle of virtual work, it is possible to demonstrate the relationship between the joint and the external loads by the well known equation:

$$\mathbf{n} = \mathbf{J}^{\mathrm{T}}\mathbf{F} \tag{1}$$

where $\mathbf{n}$ is a vector of force and moment along the joints' axes, $\mathbf{J}^{\mathrm{T}}$ is the $(6 \times L)$ transposed Jacobian matrix, $\mathbf{F}$ is (6×1) external force and moment vector acting on the manipulator's

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end-effector, and L is the number of the robot's degrees-of-freedom. It is also possible to combine the dynamic loads into this equation as shown, for example, in ref. [1].

For control purposes, where force and moment are needed only along the joints' axes, Eq. (1) is sufficient. For design purposes, however, forces acting only along the joint axes are not sufficient and the six-element force and moment vectors which also include reaction forces are needed. The aim of this investigation is to extend the dual form of the above equation in order to derive the six-element force and moment vectors at the joints. These vectors are obtained from the product of the manipulator's dual transformation matrices which constitute the generalized Jacobian matrix.

2. Derivation of the Jacobian Matrix from the Product of Dual Orthogonal Matrices

The Jacobian matrix of a robot manipulator can be explicitly obtained from the product of the manipulator's dual transformation matrices [2]. As shown below, proper columns of these products, those along the joint axes, constitute the Jacobian matrix's columns.

Consider an L degrees-of-freedom manipulator with generalized dual coordinates, $\hat{\mathbf{q}}$, i.e. $\hat{q}_i = \theta_i$ denotes a revolute joint, $\hat{q}_i = \epsilon d_i$ a prismatic joint, and $\hat{q}_i = \theta_i + \epsilon d_i$ a cylindrical joint. The velocity motor (six-element linear and angular velocity about ith coordinate system) of the ith link-attached coordinate system is:

$$\hat{\mathbf{v}}_i = {}^{i}\,\hat{\mathbf{A}}_{i-1}(\dot{\hat{\mathbf{q}}}_i + \hat{\mathbf{v}}_{i-1}) = \sum_{j=1}^{i} {}^{i}\hat{\mathbf{A}}_{j-1}\dot{\hat{\mathbf{q}}}_{\mathrm{j}}, \tag{2}$$

where the dual transformation matrix between two consecutive link-attached coordinate systems is denoted by ${}^{i-1}\hat{\mathbf{A}}_i$ and ˆ indicates, henceforth, a dual quantity.

The Jacobian matrix up to the origin of ith link attached coordinate system is given by:

$$\hat{\mathbf{J}}_i = [\hat{\mathbf{t}}_0\, \hat{\mathbf{t}}_1\, \hat{\mathbf{t}}_2, \ldots, \hat{\mathbf{t}}_{i-1}], \tag{3}$$

where $\hat{\mathbf{t}}_j$ is the column of ${}^{i}\hat{\mathbf{A}}_{j-1}$ along the direction of jth joint expressed in its local coordinate system.

3. Forces and Moments at the Joints

Eq. (1) in its dual form,

$$\hat{\mathbf{n}} = \hat{\mathbf{J}}^{\mathrm{T}}\hat{\mathbf{F}}, \tag{4}$$

projects wrenches applied at the end-effector on the joints' axes and gives the dual force—a combination of force and moment along these axes. This equation, still written in a compact three-dimensional form can be expanded to include also the dynamic loads of a robot manipulator (see [3]).

It is attractive to use the dual representation of this equation since the Jacobian matrix, **J**, can be obtained explicitly from the product of the dual transformation matrices ${}^{i}\hat{\mathbf{A}}_{i-1}$. The reason is that a dual transformation matrix contains information not only of the lines' (joint axes) directions, but also of their moments. Hence, a column corresponding to a joint axis of this matrix is exactly a corresponding column of the Jacobian matrix. This is obviously not the case with the conventional homogeneous transformation matrices, where additional computation of the moments produced by joint axes lines is needed in order to constitute the Jacobian matrix.

This investigation suggests using additional information embedded in the dual transformation matrices to calculate the complete three-dimensional dual forces (forces and moments) at the joints, and not only along the joints axes.

The transposed dual Jacobian matrix, $\hat{\mathbf{J}}^{\mathrm{T}}$, projects wrenches applied at the tool coordinate system origin onto the joints' axes direction. Since the Jacobian matrix is composed of only elements of the dual transformation matrices, the complete dual transformation matrix ${}^{i}\hat{\mathbf{A}}_{n}$ transforms a wrench from the tool coordinate system to the ith joint, and not only those projected along the joint axis. Hence, one can constitute a generalized dual Jacobian matrix as follows:

$$\hat{\mathbf{Y}} = [{}^{n}\hat{\mathbf{A}}_{0}\ {}^{n}\hat{\mathbf{A}}_{2}\ {}^{n}\hat{\mathbf{A}}_{3}, \ldots, {}^{n}\hat{\mathbf{A}}_{n-1}], \tag{5}$$

which when multiplied by the external force, $\hat{\mathbf{F}}$, yields the three-dimensional dual force $\hat{\mathbf{f}}_i$ (both force and moment) at the joints.

$$\hat{\mathbf{N}} = \hat{\mathbf{Y}}^{\mathrm{T}}\hat{\mathbf{F}}. \tag{6}$$

The generalized force vector $\hat{\mathbf{N}}$ contains all active and reactive forces at each joint:

$$\hat{\mathbf{N}} = [\hat{\mathbf{f}}_1^{\mathrm{T}}\ \hat{\mathbf{f}}_2^{\mathrm{T}}\ \hat{\mathbf{f}}_3^{\mathrm{T}}, \ldots, \hat{\mathbf{f}}_n^{\mathrm{T}}]^{\mathrm{T}}. \tag{7}$$

Next we write Eq. (6) in a form that includes also the dynamic loads. Using the virtual-work and D'Alembert principles [1] and applying the dual inertia operator [3] the following equation is obtained:

$$\hat{\mathbf{n}} = \hat{\mathbf{J}}^{\mathrm{T}}\hat{\mathbf{F}} + \sum_{i=1}^{L} \hat{\mathbf{J}}_{C_i}{}^{\mathrm{T}}[\hat{\mathbf{M}}_i[\hat{\mathbf{J}}_{C_i}\ddot{\hat{\mathbf{q}}} + \dot{\hat{\mathbf{J}}}_{C_i}\dot{\hat{\mathbf{q}}} - \hat{\mathbf{g}}] + \Omega_i\hat{\mathbf{M}}_i\hat{\mathbf{J}}_{C_i}\dot{\hat{\mathbf{q}}}], \tag{8}$$

where $\hat{\mathbf{M}}_i = [m_i\,(\mathrm{d}/\mathrm{d}\epsilon) + \epsilon\mathbf{I}_i]$ is the dual inertia operator, $\Omega_i = [\omega_i \times]$ is the angular velocity skew symmetric matrix of the ith link (Note that $\boldsymbol{\omega}_i$ is an element of $\hat{\mathbf{J}}_i\ \hat{\mathbf{q}}$), $\hat{\mathbf{J}}_{C_i} = \hat{\mathbf{D}}_i\ \hat{\mathbf{J}}_i$ is the Jacobian matrix to the center of mass of ith link, $\hat{\mathbf{D}}_i = \mathbf{1} + \epsilon[\mathbf{d}_i \times]$ is a dual matrix of translational

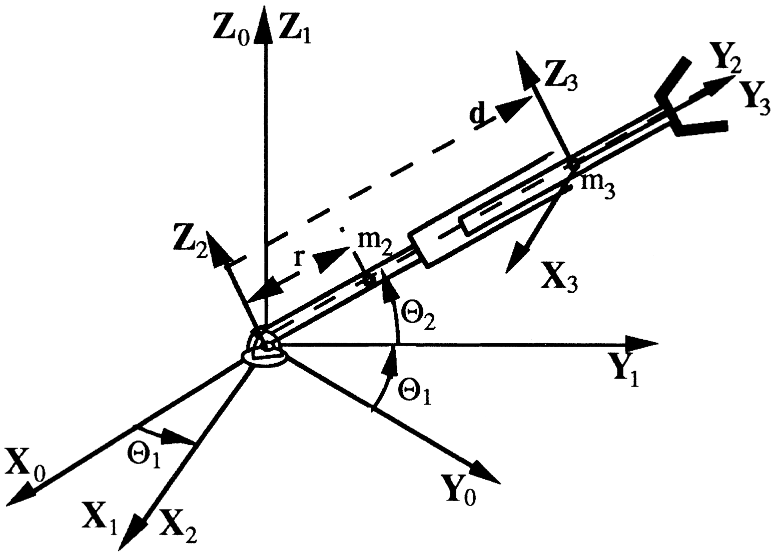

Fig. 1. Spherical three degrees-of-freedom manipulator.

transformation [4], from origin of ith coordinate system to ith link's center of mass, $[\mathbf{d}_i\times]$ is a skew symmetric (cross-product) matrix of radius-vector $\mathbf{d}_i$ from mass center i to the origin and $\mathbf{1}$ is the identity matrix.

It is worth-mentioning that Eq. (8) could be rewritten in such a form that permits calculations about arbitrary points of the links (not necessary about mass centers) by using motor derivative rules (see [5], [6]).

To obtain the full three-dimensional dual force at the joints, Eq. (6) rather than Eq. (4) is used. Hence, the robot's inverse dynamic equation is obtained by substituting the Jacobian matrices, $\hat{\mathbf{J}}_i$, with the respective generalized Jacobian, $\hat{\mathbf{Y}}_i$:

$$\hat{\mathbf{N}} = \hat{\mathbf{Y}}^{\mathrm{T}}\hat{\mathbf{F}} + \sum_{i=1}^{L} \hat{\mathbf{Y}}_{C_i}{}^{\mathrm{T}}[\hat{\mathbf{M}}_i[\hat{\mathbf{J}}_{C_i}\ddot{\mathbf{q}} + \dot{\hat{\mathbf{J}}}_{C_i}\dot{\mathbf{q}} - \hat{\mathbf{g}}] + \Omega_i\hat{\mathbf{M}}_i\hat{\mathbf{J}}_{C_i}\dot{\mathbf{q}}], \tag{9}$$

where $\hat{\mathbf{Y}}_{C_i} = \hat{\mathbf{D}}_i\hat{\mathbf{Y}}_i$ is the generalized Jacobian up to the mass center of ith link. It is worth-mentioning that the Jacobian matrices in the above equation, $\hat{\mathbf{J}}_i$, are elements of the generalized Jacobian matrices $\hat{\mathbf{Y}}_i$.

4. Illustrative Example

As an example, consider the spherical three degrees-of-freedom manipulator shown in Fig. 1

The derivation of the inverse dynamic equations of this robot requires the calculation of the Jacobian matrices to each one of the robot links center-of-mass. As mentioned above, (Section 2), the Jacobian matrix is obtained explicitly from the product of the dual transformation matrices, the third column of dual transformation matrix ${}^{3}\hat{\mathbf{A}}_0$ (rotation about z_0), the first column of ${}^{3}\hat{\mathbf{A}}_1$ (rotation about x_1) and the second column of ${}^{3}\hat{\mathbf{A}}_2$ (translation along y_2):

$$\hat{\mathbf{J}}_3 = \begin{bmatrix} -\epsilon c_2 d & 1 & 0 \\ s_2 & 0 & 1 \\ c_2 & \epsilon d & 0 \end{bmatrix}. \tag{10}$$

The additional information contained in the other columns of these dual transformation matrices is used here to derive the generalized dual Jacobian matrix.

To calculate force and moment at the joints, and not only along the joint axes, we use the generalized Jacobian matrix as defined in Eq. (5). We denote $\hat{\mathbf{Y}}_{C_i}$ as the generalized Jacobian matrix to the ith link center-of-mass. Hence the generalized Jacobian matrix to the third link expressed in its link-attached coordinate system is:

$$\hat{\mathbf{Y}}_{C_3} = [{}^{3}\hat{\mathbf{A}}_0\ {}^{3}\hat{\mathbf{A}}_1\ {}^{3}\hat{\mathbf{A}}_2] = \left[\begin{array}{ccc|ccc|ccc} c_1 - \epsilon s_1 s_2 d & s_1 + \epsilon c_1 s_2 d & -\epsilon c_2 d & 1 & \epsilon s_2 d & -\epsilon c_2 d & 1 & 0 & -\epsilon d \\ -s_1 c_2 & c_1 c_2 & s_2 & 0 & c_2 & s_2 & 0 & 1 & 0 \\ s_1 s_2 + \epsilon c_1 d & -c_1 s_2 + \epsilon s_1 d & c_2 & \epsilon d & -s_2 & c_2 & \epsilon d & 0 & 1 \end{array}\right]. \tag{11}$$

The dual generalized Jacobian to the origin of the second coordinate system is:

$$\hat{\mathbf{Y}}_2 = [{}^{2}\hat{\mathbf{A}}_0 \quad {}^{2}\hat{\mathbf{A}}_1 \quad \mathbf{0}] = \left[\begin{array}{ccc|ccc|ccc} c_1 & s_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -s_1 c_2 & c_1 c_2 & s_2 & 0 & c_2 & s_2 & 0 & 0 & 0 \\ s_1 s_2 & -c_1 s_2 & c_2 & 0 & -s_2 & c_2 & 0 & 0 & 0 \end{array}\right]. \tag{12}$$

The corresponding Jacobian matrix has a third column equal to zero since the motion of the third link has no effect on the location of the second link:

$$\hat{\mathbf{J}}_2 = \begin{bmatrix} 0 & 1 & 0 \\ s_2 & 0 & 0 \\ c_2 & 0 & 0 \end{bmatrix}, \tag{13}$$

while the Jacobian matrix up to the center of the second link is obtained by translation to the mass center:

$$\hat{\mathbf{J}}_{C_2} = \hat{\mathbf{D}}_2 \hat{\mathbf{J}}_2 = \begin{bmatrix} 1 & 0 & -\epsilon r \\ 0 & 1 & 0 \\ \epsilon r & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ s_2 & 0 & 0 \\ c_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\epsilon r c_2 & 1 & 0 \\ s_2 & 0 & 0 \\ c_2 & \epsilon r & 0 \end{bmatrix}. \quad (14)$$

Analogously, the generalized Jacobian matrix up to the center of the second link is obtained.

Assuming that the masses and inertia matrices of the second and third links are m_2 and m_3, and

$$\mathbf{I}_2 = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix} \quad \text{and} \quad \mathbf{I}_3 = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & K_3 & 0 \\ 0 & 0 & I_3 \end{bmatrix}, \quad (15)$$

one can use Eq. (9) to obtain the expression of all forces and moments at the manipulator's joints.

In order to demonstrate the approach, we show the derivation of force and moment at the second joint only. The expressions for the first and third joints are obtained in the same systematic way.

In our case, the generalized Jacobian matrices, to the center-of-mass of links 2 and 3, are:

$$\hat{\mathbf{Y}}_{c_2} = \hat{\mathbf{D}}_2 \, {}^2\hat{\mathbf{A}}_1 \quad \text{and} \quad \hat{\mathbf{Y}}_{c_3} = \hat{\mathbf{D}}_3 \, {}^3\hat{\mathbf{A}}_1 = {}^3\hat{\mathbf{A}}_1. \quad (16)$$

Substituting Eqs. (10)–(16) into Eq. (9) one obtains:

$$\begin{aligned}
\hat{\mathbf{N}} = & \begin{bmatrix} 1 & \epsilon s_2 r & -\epsilon c_2 r \\ 0 & c_2 & s_2 \\ \epsilon r & -s_2 & c_2 \end{bmatrix}^{\mathrm{T}} \left\{ \hat{\mathbf{M}}_2 \left[\begin{bmatrix} -\epsilon c_2 r & 1 & 0 \\ s_2 & 0 & 0 \\ c_2 & \epsilon r & 0 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{d} \end{bmatrix} + \begin{bmatrix} \epsilon s_2 r \dot{\theta}_2 & 0 & 0 \\ c_2 \dot{\theta}_2 & 0 & 0 \\ -s_2 \dot{\theta}_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d} \end{bmatrix} \right. \right. \\
& \left. \left. - \begin{bmatrix} 0 \\ -\epsilon g s_2 \\ -\epsilon g c_2 \end{bmatrix} \right] + \begin{bmatrix} 0 & -\dot{\theta}_1 c_2 & \dot{\theta}_1 s_2 \\ \dot{\theta}_1 c_2 & 0 & -\dot{\theta}_2 \\ -\dot{\theta}_1 s_2 & \dot{\theta}_2 & 0 \end{bmatrix} \hat{\mathbf{M}}_2 \begin{bmatrix} -\epsilon c_2 r & 1 & 0 \\ s_2 & 0 & 0 \\ c_2 & \epsilon r & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d} \end{bmatrix} \right\} \\
& + \begin{bmatrix} 1 & \epsilon s_2 d & -\epsilon c_2 d \\ 0 & c_2 & s_2 \\ \epsilon d & -s_2 & c_2 \end{bmatrix}^{\mathrm{T}} \left\{ \hat{\mathbf{M}}_3 \left[\begin{bmatrix} -\epsilon c_2 d & 1 & 0 \\ s_2 & 0 & 1 \\ c_2 & \epsilon d & 0 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{d} \end{bmatrix} + \begin{bmatrix} \epsilon(s_2 d \dot{\theta}_2 - c_2 \dot{d}) & 0 & 0 \\ c_2 \dot{\theta}_2 & 0 & 0 \\ -s_2 \dot{\theta}_2 & \epsilon \dot{d} & 0 \end{bmatrix} \right. \right. \\
& \left. \left. \times \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d} \end{bmatrix} - \begin{bmatrix} 0 \\ -\epsilon g s_2 \\ -\epsilon g c_2 \end{bmatrix} \right] + \begin{bmatrix} 0 & -\dot{\theta}_1 c_2 & \dot{\theta}_1 s_2 \\ \dot{\theta}_1 c_2 & 0 & -\dot{\theta}_2 \\ -\dot{\theta}_1 s_2 & \dot{\theta}_2 & 0 \end{bmatrix} \hat{\mathbf{M}}_3 \begin{bmatrix} -\epsilon c_2 d & 1 & 0 \\ s_2 & 0 & 1 \\ c_2 & \epsilon d & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d} \end{bmatrix} \right\} \quad (17)
\end{aligned}$$

After rearranging terms, one finally has:

$$\hat{\mathbf{N}} = \begin{bmatrix} (m_2r + m_3d)(2s_2\dot{\theta}_1\dot{\theta}_2 - c_2\ddot{\theta}_1) - 2m_3c_2\dot{d}\dot{\theta}_1 + \epsilon[(m_2r^2 + m_3d^2)(\ddot{\theta}_2 + c_2s_2\dot{\theta}_1^2) \\ +(m_2r + m_3d)c_2g + 2m_3d\dot{d}\dot{\theta}_2 + I_{23}\ddot{\theta}_2 + (I_{23} - K_{23})c_2s_2\dot{\theta}_1^2] \\ m_3(\ddot{d}c_2 - 2s_2\dot{d}\dot{\theta}_2) - (m_2r + m_3d)(c_2\dot{\theta}_1^2 + c_2\dot{\theta}_2^2 + s_2\ddot{\theta}_2) + \epsilon[(I_{23} - K_{23})(s_2^2\dot{\theta}_1\dot{\theta}_2 - c_2s_2\ddot{\theta}_1) \\ -2m_3dc_2s_2\dot{d}\dot{\theta}_1 + (m_2r^2 + m_3d^2)(2s_2^2\dot{\theta}_1\dot{\theta}_2 - c_2s_2\ddot{\theta}_1) + (I_{23}s_2^2 + K_{23}c_2^2)\dot{\theta}_1\dot{\theta}_2] \\ m_3(\ddot{d}s_2 + 2c_2\dot{d}\dot{\theta}_2) - (m_2r + m_3d)(c_2\ddot{\theta}_2 - s_2\dot{\theta}_2^2) + (m_2 + m_3)g + \epsilon[2m_3dc_2^2\dot{d}\dot{\theta}_1 + \\ (m_2r^2 + m_3d^2)(c_2^2\ddot{\theta}_1 - 2c_2s_2\dot{\theta}_1\dot{\theta}_2) - 2(I_{23} - K_{23})c_2s_2\dot{\theta}_1\dot{\theta}_2 + (I_{23}c_2^2 + K_{23}s_2^2)\ddot{\theta}_1] \end{bmatrix}, \tag{18}$$

where $I_{23} = I_2 + I_3$ and $K_{23} = K_2 + K_3$.

Eq. (18) contains all forces and moments at the origin of the second link-attached coordinate system. Obviously, moment along the second actuator, as obtained by the traditional use of Eq. (1), is included in the above expression, dual part of the X (first) component.

5. Conclusions

The use of dual numbers in robot kinematics allows one to obtain the Jacobian matrix directly from the product of dual transformation matrices without any additional computations. In this investigation, we used additional information contained in the dual transformation matrices to compose the generalized Jacobian matrix with no additional computations as well. Using the complete dual transformation matrix, one obtains the generalized Jacobian matrix from which forces and moment at each joint in all directions (not only those along the joint axes) are derived. Although the Jacobian matrix is sufficient for control purposes where only force and moment along the joint axes are needed, for design purposes all components of force and moment are required. These components are derived from the generalized Jacobian matrix.

In dynamics, the Newton–Euler formulation has been long used to derive forces and moments at the robot's joints (this was usually used to derive the inverse dynamics which require only the force and moment along the joint axes). The present scheme gives the same result. It is, however, expressed almost exclusively in terms of the dual generalized Jacobian matrix which is obtained with no further computation from the product of the dual transformation matrices.

It should be mentioned that the dual generalized Jacobian matrix is the expression that specifically provides the relationship between small motions in all directions at the joints and small motion of the end-effector. This is useful at the design phase of robot manipulators - especially flexible ones - where the designer takes into consideration motions at the joints in all directions, not only along the joint axes.

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