Geometric Interpretation of the Derivatives of Parallel Robots' Jacobian Matrix with Application to Stiffness Control

N. Simaan and M. Shoham
Robotics Laboratory
Department of Mechanical Engineering
Technion – Israel Institute of Technology
Haifa 32000, Israel
nabil@tx.technion.ac.il
shoham@tx.technion.ac.il

ABSTRACT

This paper presents a closed-form formulation and geometrical interpretation of the derivatives of the Jacobian matrix of fully parallel robots with respect to the moving platforms' position/orientation variables. Similar to the Jacobian matrix, these derivatives are proven to be also groups of lines that together with the lines of the instantaneous direct kinematics matrix govern the singularities of the active stiffness control problem. This geometric interpretation is utilized in an example of a planar 3 degrees-of-freedom redundant robot to determine its active stiffness control singularity.

1. INTRODUCTION

Line geometry has been applied by several researchers to the kinematics and statics of parallel manipulators (Merlet, 1989; Colling and Long, 1995; Ben Horin, 1997; Simaan, 1999 and 2001; Pottman, Peternell, and Ravani, 1999). Line geometry is used because the rows of the Jacobian matrix in a linearly actuated fully-parallel manipulator are the Plücker line coordinates of the axes of its extensible links (Hunt, Samuel, and McAree, 1991). Hence, linear dependence of these lines determines the conditions for instability and singularity of a parallel manipulator as Dandurand (1984) has shown in the context of stability of spatial grids.

In contrast to the numerous investigations devoted to the formulation of parallel manipulators' Jacobian matrix e.g., (Cleary and Uebel, 1994; Simaan, Glozman, and Shoham, 1998; Tsai, 1998-1999), there are only a few studies addressing the formulation of its derivative. Dutré, et al., (1997) addressed this problem and obtained a closed form analytic expression for the derivative of the inverse Jacobian matrix with respect to time and with respect to the joint active variables. Merlet and Gosselin (1991) formulated the time derivative of the Jacobian of a fully parallel manipulator for use in acceleration analysis.

Duffy (1996) presented the infinitesimal motion and stiffness analysis of a planar parallel manipulator and obtained a stiffness matrix of the manipulator with a preloaded spring model. He showed that the part of the stiffness matrix that corresponds to the preload effect is a product of two matrices having line-coordinates as their columns.

This paper is organized as the following: the first part, sections 2 and 3, formulates the derivatives of the Jacobian matrix with respect to the moving platform position/orientation variables and associates a geometric interpretation to these derivatives as groups of lines. These derivatives play a major role in stiffness analysis and control (modulation) (Yi, Freeman, and Tesar 1989; Kock and Schumacher, 1998), dynamic manipulability analysis (Yoshikawa, 1990), and force-controlled compliant motions (Dutré, Bruyninckx, and Schutter, 1997). The second part, section 4 emphasizes the contribution of these derivatives to manipulator's rigidity and active stiffness control and relates each one of these Jacobian derivatives with a direction of the controlled stiffness. Section 5 relates singularities of the Jacobian derivatives with singularities of the stiffness control scheme and singularities of the derivatives of the instantaneous direct kinematics matrix, **A**, presented in the next section. Section 6 shows that the stiffness modulation singularities can be found using line geometry tools as a result of the line-based interpretation of the Jacobian derivatives and the instantaneous direct kinematics matrix. Finally, based on line geometry of the derivatives of its instantaneous direct kinematics matrix, an example of a planar 3 degrees-of-freedom redundant parallel robot is presented in a stiffness modulation singular position.

2 BACKGROUND - JACOBIAN FORMULATION

Consider a general Stewart-Gough type parallel manipulator subject to a wrench $\mathbf{F}_{\mathrm{env}} = \left[\mathbf{f}_{\mathrm{env}}{}^{T}, \mathbf{m}_{\mathrm{env}}{}^{T}\right]^{T}$ applied by the environment, Fig. 1. Let the position/orientation vector of the moving platform relative to world coordinate system be $\mathbf{X} = \left[x, y, z, \theta_x, \theta_y, \theta_z\right]^{T}$, where x, y, z are the Cartesian coordinates and θ_x, θ_y, and θ_z are the orientation angles of the moving platform, and let $\dot{\mathbf{x}}$ denote the end effector twist and $\dot{\mathbf{q}}$ the corresponding active joints' rates.

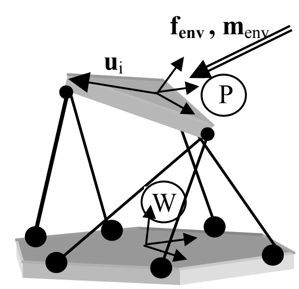

Figure 1: Typical Stewart-Gough Manipulator

For parallel manipulators, the commonly used expression of the Jacobian matrix is:

$$\dot{\mathbf{q}} = \mathbf{J}\dot{\mathbf{x}}\,, \quad \left(\mathrm{J}_{ij} = \frac{\partial \mathrm{q}_i}{\partial \mathrm{x}_j} \right) \tag{1}$$

which is the inverse of that of serial manipulators': $\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}\,, \quad \left(\mathrm{J}_{ij} = \frac{\partial \mathrm{x}_i}{\partial \mathrm{q}_j} \right)$.

In this paper we use the Jacobian, $\mathbf{J}$, in Eq. (1) to map the end effector twist, $\dot{\mathbf{x}}$, to active joint rates, $\dot{\mathbf{q}}$. This Jacobian matrix is also used to relate the required active joints' forces for a desired external wrench $\mathbf{F}_e = \left[\mathbf{f}_e^{\mathrm{T}}, \mathbf{m}_e^{\mathrm{T}}\right]^{\mathrm{T}}$ to be exerted on the environment $\left(\mathbf{F}_e = -\mathbf{F}_{env}\right)$.

$$\mathbf{J}^{\mathrm{T}}\boldsymbol{\tau} = \mathbf{F}_e \tag{2}$$

Using the loop closure method (Ma and Angeles, 1992), or the static equilibrium method (Cleary and Uebel, 1994; Simaan, Glozman, and Shoham, 1998; Simaan, 1999), along with Eqs. (1) and (2), respectively, yields the commonly used formulation of the Jacobian matrix.

$$\mathbf{J} = \begin{bmatrix} \hat{\mathbf{l}}_1^{\mathrm{T}} & \left({}^{w}\mathbf{R}_p \mathbf{u}_1 \times \hat{\mathbf{l}}_1\right)^{\mathrm{T}} \\ \vdots & \vdots \quad \vdots \\ \hat{\mathbf{l}}_6^{\mathrm{T}} & \left({}^{w}\mathbf{R}_p \mathbf{u}_6 \times \hat{\mathbf{l}}_6\right)^{\mathrm{T}} \end{bmatrix} \tag{3}$$

where $\hat{\mathbf{l}}_i$ denotes a unit vector along the i^{th} active prismatic joint pointing from its joint at the base to its joint at the moving platform. The platform-attached and the base-attached coordinate systems are referred to by the letters P and W, respectively, Fig. 1. Accordingly, ${}^{w}\mathbf{R}_p$ is the rotation matrix from P to W, and $\mathbf{u}_i$ is the constant position vector of the i^{th} joint in P, Fig. 1.

In order to interpret the Jacobian matrix as lines, the following basic definitions of line geometry are reviewed. A given sextuplet of numbers $\left[\mathrm{l}_{vx}, \mathrm{l}_{vy}, \mathrm{l}_{vz}, \mathrm{l}_{mx}, \mathrm{l}_{my}, \mathrm{l}_{mz}\right]^{\mathrm{T}}$ represents a line in space only when it belongs to a five-dimensional quadratic manifold called the Grassmannian (Merlet, 1989; Pellegrini, 1997), the Plücker hypersurface (Graustein 1930, Sommerville, 1934) or Klein quadric (Pottman, Peternell, and Ravani, 1999; Pellegrini, 1997) or, in other words, it fulfils Eq. (4).

$$\mathrm{l}_{vx}\,\mathrm{l}_{mx} + \mathrm{l}_{vy}\,\mathrm{l}_{my} + \mathrm{l}_{vz}\,\mathrm{l}_{mz} = 0 \tag{4}$$

Observing Eq. (3), it is clear that the rows of the Jacobian are the Plücker ray coordinates of lines along the prismatic actuators. This geometrical interpretation is correct in a coordinate system having its origin attached to the moving platform in its center. In this representation each row of the Jacobian matrix is a function of ${}^{W}\mathbf{R}_p\mathbf{u}_i$ and the direction numbers of $\hat{\mathbf{l}}_i$, which are both functions of the moving platform position.

2.1 The lines of the Jacobian matrix in world coordinate system

Consider another representation of the Jacobian matrix in the form:

$$\mathbf{J}_b{}^T\tau = \mathbf{F}_b \tag{5}$$

where $\mathbf{F}_b = \left[\mathbf{f}_b{}^T, \mathbf{m}_b{}^T\right]^T$ represents the wrench exerted by the base rather than the moving platform on the environment, Fig 2. By using simple statics equations and representing $\mathbf{F}_b$ by $\mathbf{F}_e$ one obtains:

$$\mathbf{A}\tau = \mathbf{B}\mathbf{F}_e \tag{6}$$

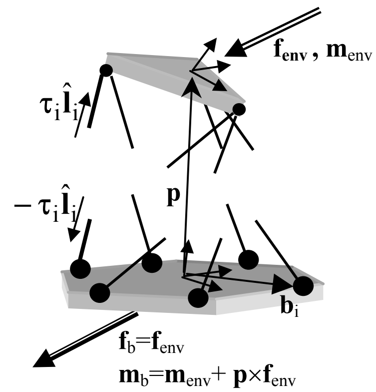

Figure 2: Static equilibrium on base and moving platform

where: $$\mathbf{A} = \begin{bmatrix} \hat{\mathbf{l}}_1 & \cdots & \hat{\mathbf{l}}_6 \\ \mathbf{b}_1\times\hat{\mathbf{l}}_1 & \cdots & \mathbf{b}_6\times\hat{\mathbf{l}}_6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ [\mathbf{p}\times] & \mathbf{I} \end{bmatrix} \tag{7}$$

$\mathbf{I}$ – 3×3 unit matrix

$\mathbf{b}_i$ – position vector of the spherical joint of the i^{th} prismatic actuator at the base in W coordinate system.

$[\mathbf{p}\times]$ – skew-symmetric matrix representing vector multiplication.

$$[\mathbf{p}\times] = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix} \tag{8}$$

Eqs. (5) and (6) yield: $$\mathbf{J}^T = \mathbf{B}^{-1}\mathbf{A} \tag{9}$$

Where $\mathbf{B}^{-1}$ is given by: $$\mathbf{B}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ [-\mathbf{p}\times] & \mathbf{I} \end{bmatrix} \tag{10}$$

Contrary to ${}^{W}\mathbf{R}_p\mathbf{u}_i$, which is a varying vector in W, the vector $\mathbf{b}_i$ is constant in W. This simplifies the expression of the derivative of $\mathbf{J}^T$. In this formulation, the lines of $\mathbf{A}$ pass through fixed points, $\mathbf{b}_i$, in W and therefore their derivatives are easily shown to be lines as will be shown later.

The physical interpretation of multiplying a Plücker line's coordinates by the matrix $\mathbf{B}^{-1}$ is a translation the line while maintaining its direction. Figure 3 shows a 6–6 Stewart-Gough platform manipulator with the lines of the

Jacobian in W indicated by thin arrows. Another important feature of $\mathbf{B}^{-1}$ is that its determinant is equal to 1, which means that the above multiplication, Eq. (9), does not add to the singularities of **J**.

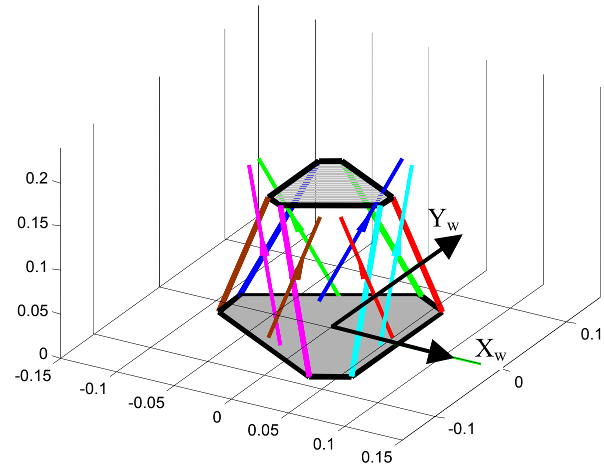

Figure 3: Lines of the Jacobian in W (world coordinate system)

3 FORMULATION OF THE DERIVATIVE OF THE JACOBIAN MATRIX

The derivatives of $\mathbf{J}^t$ with respect to the moving platform's position variables is obtained from Eq. (9) as:

$$\frac{d\mathbf{J}^T}{d\mathbf{x}} = \frac{d\mathbf{B}^{-1}}{d\mathbf{x}}\mathbf{A} + \mathbf{B}^{-1}\frac{d\mathbf{A}}{d\mathbf{x}} \tag{11}$$

The matrices $\frac{d\mathbf{J}^T}{d\mathbf{x}}$, $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}$, $\frac{d\mathbf{A}}{d\mathbf{x}}$ are three-dimensional 6×6×6 matrices for non-redundant six degrees-of-freedom manipulators. The i^{th} plane of these matrices is their derivative with respect to the i^{th} position/orientation coordinate, x_i, of the moving platform.

The multiplication in Eq. (11) is performed plane by plane, i.e., the derivative of $\mathbf{J}^t$ with respect to the i^{th} position/orientation variable is obtained by multiplying the i^{th} planes of $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}$ and $\frac{d\mathbf{A}}{d\mathbf{x}}$ with **A** and $\mathbf{B}^{-1}$, respectively.

The derivative of $\mathbf{B}^{-1}$ is simple and yields a matrix whose structure is similar to $\mathbf{B}^{-1}$ so the first expression on the right hand side of Eq. (11) yields a matrix whose columns are the translated lines of **A** under the transformation $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}$. If the derivative $\frac{d\mathbf{A}}{d\mathbf{x}}$ yields a matrix whose columns are also lines and the translated lines $\mathbf{B}^{-1}\frac{d\mathbf{A}}{d\mathbf{x}}$ intersect the lines of $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}\mathbf{A}$, then the derivative of **J** is also a matrix with lines as its columns. This is true since any linear combination of two given intersecting lines spans a flat pencil of lines (Graustein, 1930).

3.1 Derivative of the matrix A

The matrix **A** in Eq. (7) is composed of the lines along the robot's prismatic joints. Each unit vector along these lines is characterized by its direction cosines α_i, β_i, and γ_i, Eq. (12).

$$\hat{\mathbf{l}}_i = \left[\cos(\alpha_i), \cos(\beta_i), \cos(\gamma_i)\right]^T \tag{12}$$

The matrix $\frac{d\mathbf{A}}{d\mathbf{x}}$ is a three-dimensional 6×6×6 matrix with the i^{th} plane, $\frac{\partial \mathbf{A}}{\partial \mathbf{x}_i}$, being the derivative of **A** with respect to the i^{th} position/orientation coordinate, x_i, of the moving platform. Since **A** has the lines $\mathbf{l}_i$ as its columns, we are interested in finding the derivatives of these lines.

Using Eq. (7) while keeping in mind that the vectors $\mathbf{b}_i$ are constant one can write:

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where
$$\frac{\partial \mathbf{l}_j}{\partial x_i} = \frac{\partial \mathbf{l}_j}{\partial \alpha_j}\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \mathbf{l}_j}{\partial \beta_j}\frac{\partial \beta_j}{\partial x_i} + \frac{\partial \mathbf{l}_j}{\partial \gamma_j}\frac{\partial \gamma_j}{\partial x_i} \tag{14}$$

In order to write Eq. (14) in a matrix form, we define three matrices $\frac{d\mathbf{A}}{d\alpha}$, $\frac{d\mathbf{A}}{d\beta}$, and $\frac{d\mathbf{A}}{d\gamma}$:

$$\frac{\partial \mathbf{A}}{\partial \alpha} = \begin{bmatrix} \frac{\partial \mathbf{l}_1}{\partial \alpha_1} & \cdots & \frac{\partial \mathbf{l}_n}{\partial \alpha_n} \end{bmatrix} \qquad \frac{\partial \mathbf{A}}{\partial \beta} = \begin{bmatrix} \frac{\partial \mathbf{l}_1}{\partial \beta_1} & \cdots & \frac{\partial \mathbf{l}_n}{\partial \beta_n} \end{bmatrix} \qquad \frac{\partial \mathbf{A}}{\partial \gamma} = \begin{bmatrix} \frac{\partial \mathbf{l}_1}{\partial \gamma_1} & \cdots & \frac{\partial \mathbf{l}_n}{\partial \gamma_n} \end{bmatrix} \tag{15}$$

We also define $\mathbf{J}_{d\alpha i}, \mathbf{J}_{d\beta i}, \mathbf{J}_{d\gamma i}$ as three diagonal matrices having on their main diagonals the i^{th} columns of $\mathbf{J}_\alpha$, $\mathbf{J}_\beta$, and $\mathbf{J}_\gamma$ respectively, where $\mathbf{J}_\alpha$, $\mathbf{J}_\beta$, and $\mathbf{J}_\gamma$ are given by:

$$J_{\alpha_{m,n}} = \frac{\partial \alpha_m}{\partial x_n}, \qquad J_{\beta_{m,n}} = \frac{\partial \beta_m}{\partial x_n}, \qquad J_{\gamma_{m,n}} = \frac{\partial \gamma_m}{\partial x_n} \tag{16}$$

Using these definitions one can write Eq. (13) in matrix form as:

$$\frac{\partial \mathbf{A}}{\partial x_i} = \frac{\partial \mathbf{A}}{\partial \alpha}\mathbf{J}_{d_{\alpha i}} + \frac{\partial \mathbf{A}}{\partial \beta}\mathbf{J}_{d_{\beta i}} + \frac{\partial \mathbf{A}}{\partial \gamma}\mathbf{J}_{d_{\gamma i}} \tag{17}$$

The derivatives of the lines with respect to their variables (keeping in mind that $\mathbf{b}_i$ are constant) are:

$$\frac{\partial \mathbf{l}_i}{\partial \alpha_i} = \left[\mathbf{l}_{v\alpha}{}^T \ \mathbf{l}_{m\alpha}{}^T\right]^T \qquad \mathbf{l}_{v\alpha} = \left[-\sin(\alpha_i), 0, 0\right]^T \quad \mathbf{l}_{m\alpha} = \left[\mathbf{b}_i \times \left(-\sin(\alpha_i), 0, 0\right)\right]^T \tag{18}$$

$$\frac{\partial \mathbf{l}_i}{\partial \beta_i} = \left[\mathbf{l}_{v\beta}{}^T \ \mathbf{l}_{m\beta}{}^T\right]^T \qquad \mathbf{l}_{v\beta} = \left[0, -\sin(\beta_i), 0\right]^T \quad \mathbf{l}_{m\beta} = \left[\mathbf{b}_i \times \left(0, -\sin(\beta_i), 0\right)\right]^T \tag{19}$$

$$\frac{\partial \mathbf{l}_i}{\partial \gamma_i} = \left[\mathbf{l}_{v\gamma}{}^T \ \mathbf{l}_{m\gamma}{}^T\right]^T \qquad \mathbf{l}_{v\gamma} = \left[0, 0, -\sin(\gamma_i)\right]^T \quad \mathbf{l}_{m\gamma} = \left[\mathbf{b}_i \times \left(0, 0, -\sin(\gamma_i)\right)\right]^T \tag{20}$$

It can be seen that Eqs. (18-20) are also lines that intersect the lines of the matrix **A** at points $\mathbf{b}_i$.

For each line $\hat{\mathbf{l}}_i$, the direction cosines are related by Eq. (21):

$$\cos(\alpha_i)^2 + \cos(\beta_i)^2 + \cos(\gamma_i)^2 = 1 \tag{21}$$

Differentiating Eq. (21) with respect to x_i and solving for $\frac{\partial \gamma_i}{\partial x_i}$ yields:

$$\frac{\partial \gamma_i}{\partial x_i} = \frac{-c_{\alpha_i} s_{\alpha_i}}{c_{\gamma_i} s_{\gamma_i}} \frac{\partial \alpha_i}{\partial x_i} + \frac{-c_{\beta_i} s_{\beta_i}}{c_{\gamma_i} s_{\gamma_i}} \frac{\partial \beta_i}{\partial x_i} \tag{22}$$

Where the abbreviations s_α and c_α stand for sin α and cos α respectively.

Substituting Eq. (22) in (14) and eliminating $\frac{\partial \gamma_i}{\partial x_j}$ yields:

$$\begin{gathered}\frac{\partial \mathbf{l}_i}{\partial x_j} = \frac{\partial \alpha_i}{\partial x_j}\left[\mathbf{p}_i^{\,T} \ \ (\mathbf{b}_i \times \mathbf{p}_i)^T\right]^T + \frac{\partial \beta_i}{\partial x_j}\left[\mathbf{r}_i^{\,T} \ \ (\mathbf{b}_i \times \mathbf{r}_i)^T\right]^T \\ \mathbf{p}_i = \left[-s_{\alpha i},\, 0,\, \frac{c_{\alpha_i} s_{\alpha_i}}{c_{\gamma_i}}\right]^T \quad \mathbf{r}_i = \left[0,\, -s_{\beta_i},\, \frac{c_{\beta_i} s_{\beta_i}}{c_{\gamma_i}}\right]^T\end{gathered} \tag{23}$$

The first and the second brackets in the expression of $\frac{\partial \mathbf{l}_i}{\partial x_j}$ in Eq. (23) are the 6×1 column vectors $\frac{\partial \mathbf{l}_i}{\partial \alpha_i}$ and $\frac{\partial \mathbf{l}_i}{\partial \beta_i}$, respectively. Both these brackets represent lines according to Eq. (4) and it is easy to see that both lines are perpendicular to $\mathbf{l}_i$. The expressions $\frac{\partial \alpha_i}{\partial x_j}$ and $\frac{\partial \beta_i}{\partial x_j}$ are scalars. Consequently, the columns of $\frac{\partial \mathbf{A}}{\partial x_i}$ in Eq. (13) are lines that pass through the spherical joints in the points $\mathbf{b}_i$ and belong to the flat pencils of $\frac{\partial \mathbf{l}_i}{\partial \alpha_i}$ and $\frac{\partial \mathbf{l}_i}{\partial \beta_i}$. This interpretation will prove to be helpful in section 6 where geometric interpretation to the stiffness modulation singularities is sought.

Summarizing this section, we conclude that the lines of the derivative of **A** are perpendicular to the lines of **A** and intersect them in the points $\mathbf{b}_i$, i.e., in the spherical joints at the base platform. This fact is used to show that the derivative of the Jacobian matrix, Eq. (11), is also a group of lines.

3.2 Explicit expressions of J_α, J_β, and J_γ

The explicit expressions of J_α, J_β, and J_γ which constitute the derivative of **A,** Eqs. (17) and (16), are developed below. Figure 4 depicts a fully-parallel robot with six independent closed loops. Each loop is governed by the loop equation:

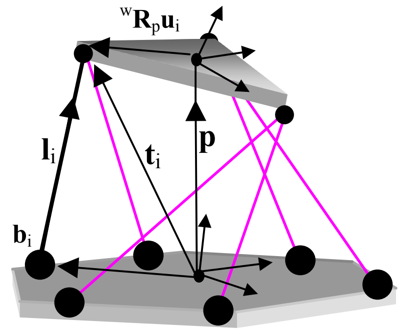


Figure 4: Kinematic closed loops

$$\mathbf{p}+{}^{w}\mathbf{R}_p\mathbf{u}_i = \mathbf{b}_i + q_i\hat{\mathbf{l}}_i \tag{24}$$

where q_i represents the length of the i^{th} prismatic joint, $\mathbf{p}$ the position of the moving platform in W, and ${}^{w}\boldsymbol{\omega}^{p}$ the angular velocity of the moving platform in W.

Taking the time derivative of Eq. (24) yields:

$$\dot{\mathbf{p}}-{}^{w}\mathbf{R}_p\mathbf{u}_i\times{}^{w}\boldsymbol{\omega}^{p} = \dot{q}_i\hat{\mathbf{l}}_i + q_i\dot{\hat{\mathbf{l}}}_i \tag{25}$$

Rewriting the right-hand side of Eq. (25) in terms of the vector of linear/angular velocities of the moving platform, $\dot{\mathbf{x}} = \left[\begin{array}{cc}\dot{\mathbf{p}}^{T} & \left({}^{w}\boldsymbol{\omega}^{p}\right)^{T}\end{array}\right]^{T}$, yields:

$$\dot{\mathbf{p}}-{}^{w}\mathbf{R}_p\mathbf{u}_i\times{}^{w}\boldsymbol{\omega}^{p} = \left[\mathbf{I} \quad ,\left[-\left({}^{w}\mathbf{R}_p\mathbf{u}_i\right)\times\right]\right]\dot{\mathbf{x}} \equiv \mathbf{M}_i\dot{\mathbf{x}} \tag{26}$$

Expression $\dot{\mathbf{q}}_i\hat{\mathbf{l}}_i$ in Eq. (25) is expressed in terms of $\dot{\mathbf{x}}$ by using the velocity relation $\dot{\mathbf{q}} = \mathbf{J}\dot{\mathbf{x}}$ with reference to the i^{th} row of $\mathbf{J}$ as $\mathbf{J}_i$, and using Eq. (12) for $\hat{\mathbf{l}}_i$:

$$\dot{q}_i\hat{\mathbf{l}}_i = \begin{bmatrix}\cos(\alpha_i)\mathbf{J}_i\\ \cos(\beta_i)\mathbf{J}_i\\ \cos(\gamma_i)\mathbf{J}_i\end{bmatrix}\dot{\mathbf{x}} \equiv \mathbf{N}_i\dot{\mathbf{x}} \qquad \mathbf{N}_i \in \Re^{3\times 6} \tag{27}$$

Substituting back into Eq. (25) yields:

$$q_i\begin{bmatrix}-\sin(\alpha_i)\dot{\alpha}_i\\ -\sin(\beta_i)\dot{\beta}_i\\ -\sin(\gamma_i)\dot{\gamma}_i\end{bmatrix} = \left[\mathbf{M}_i - \mathbf{N}_i\right]\dot{\mathbf{x}} \qquad \mathbf{M}_i, \mathbf{N}_i \in \Re^{3\times 6} \tag{28}$$

Solving Eq. (28) for its unknowns $\dot{\alpha}_i, \dot{\beta}_i$, and $\dot{\gamma}_i$ yields:

$$\dot{\alpha}_i = \left[\frac{-1}{q_i\sin(\alpha_i)}\left[\mathbf{M}_i - \mathbf{N}_i\right]_1\right]\dot{\mathbf{x}}, \qquad \dot{\beta}_i = \left[\frac{-1}{q_i\sin(\beta_i)}\left[\mathbf{M}_i - \mathbf{N}_i\right]_2\right]\dot{\mathbf{x}}, \qquad \dot{\gamma}_i = \left[\frac{-1}{q_i\sin(\gamma_i)}\left[\mathbf{M}_i - \mathbf{N}_i\right]_3\right]\dot{\mathbf{x}} \tag{29}$$

Where $[\mathbf{M}_i\text{-}\mathbf{N}_i]_j$ is the j^{th} row of $[\mathbf{M}_i\text{-}\mathbf{N}_i]$, j = 1, 2, 3.

Equation (30) gives the i^{th} rows of $\mathbf{J}_\alpha$, $\mathbf{J}_\beta$, and $\mathbf{J}_\gamma$ as:

$$[\mathbf{J}_\alpha]_i=\left[\frac{-1}{q_i \sin(\alpha_i)}[\mathbf{M}_i-\mathbf{N}_i]_1\right], \qquad [\mathbf{J}_\beta]_i=\left[\frac{-1}{q_i \sin(\beta_i)}[\mathbf{M}_i-\mathbf{N}_i]_2\right], \qquad [\mathbf{J}_\gamma]_i=\left[\frac{-1}{q_i \sin(\gamma_i)}[\mathbf{M}_i-\mathbf{N}_i]_3\right] \tag{30}$$

This completes the formulation of the necessary terms in Eq. (17) and, thus, the derivative of **A** is fully defined and proven to be a matrix whose columns are lines. These lines are perpendicular to the lines of **A** and intersect them at the spherical joints at the base, $\mathbf{b}_i$. What remains is to show that the sum of the terms in Eq. (11) gives a set of lines.

3.3 Intersection of the lines of $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}_i}\mathbf{A}$ and the lines of $\mathbf{B}^{-1}\frac{d\mathbf{A}}{d\mathbf{x}_i}$

Recalling the definition of **X** and matrix **B** (section 2) and observing Eq. (11), one concludes that the last three planes of $\frac{d\mathbf{J}}{dx}$, i.e. $\frac{\partial \mathbf{J}}{\partial x_k}$ (k=4,5,6) are the translated lines of $\frac{\partial \mathbf{A}}{\partial \mathbf{x}_k}$ (k=4,5,6) under the transformation $\mathbf{B}^{-1}$.

This can be written as:

$$\frac{\partial \mathbf{J}^t}{\partial \mathbf{x}_i}=\mathbf{B}^{-1}\frac{\partial \mathbf{A}}{\partial \mathbf{x}_i} \qquad i=4,5,6. \tag{31}$$

It remains to prove that the derivatives with respect to the Cartesian coordinates, $\frac{\partial \mathbf{J}}{\partial x_i}$ for i = 1, 2, 3, represent lines.

In order to prove this, one must prove that the lines of $\frac{\partial \mathbf{B}^{-1}}{\partial \mathbf{x}_i}\mathbf{A}$ intersect the lines of $\mathbf{B}^{-1}\frac{\partial \mathbf{A}}{\partial \mathbf{x}_i}$.

The following proof relies on the condition of intersection between two given lines, $\mathbf{l} = [l_1, l_2, l_3, l_4, l_5, l_6]^t$ and $\mathbf{m} = [m_1, m_2, m_3, m_4, m_5, m_6]^t$. This condition is given in Eq. (32) and has the interpretation of the moment of a force acting along line **l** about line **m** (Hunt, 1978).

$$l_1m_4 + l_2m_5 + l_3m_6 + l_4m_1 + l_5m_2 + l_6m_1 = 0 \tag{32}$$

This is proven symbolically using Maple® (a symbolic manipulation program) and also verified numerically with a numerical and a graphical simulation using Matlab®.

The i^{th} column of **A** and i^{th} row of **J** are given by Eq. (33). The i^{th} rows of $\mathbf{J}_\alpha$, $\mathbf{J}_\beta$, and $\mathbf{J}_\gamma$ are given in Appendix-A1.

$$\begin{aligned}\mathbf{J}_i &= [c_{\alpha_i}, c_{\beta_i}, c_{\gamma_i}, p_z c_{\beta_i} - p_y c_{\gamma_i} + b_{i_y} c_{\gamma_i} - b_{i_z} c_{\beta_i}, -p_z c_{\alpha_i} + p_x c_{\gamma_i} + b_{i_z} c_{\alpha_i} - b_{i_x} c_{\gamma_i}, p_y c_{\alpha_i} - p_x c_{\beta_i} + b_{i_x} c_{\beta_i} - b_{i_y} c_{\alpha_i}] \\ \mathbf{A}^i &= [c_{\alpha_i}, c_{\beta_i}, c_{\gamma_i}, b_{i_y} c_{\gamma_i} - b_{i_z} c_{\beta_i}, b_{i_z} c_{\alpha_i} - b_{i_x} c_{\gamma_i}, b_{i_x} c_{\beta_i} - b_{i_y} c_{\alpha_i}]\end{aligned} \tag{33}$$

In the following sub-sections we formulate the derivatives $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}\mathbf{A}$ and $\mathbf{B}^{-1}\frac{\partial \mathbf{A}}{\partial \mathbf{x}_i}$. The resulting expressions are used in Eq. (32) to complete the proof that the derivatives of the Jacobian are lines.

3.3.1 Formulation of $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}\mathbf{A}$

The derivatives of $\mathbf{B}^{-1}$ are simple and can be written as:

$$\frac{\partial \mathbf{B}^{-1}}{\partial x_i} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \frac{\partial([\mathbf{p}\times])}{\partial x_i} & \mathbf{0} \end{bmatrix} \tag{34}$$

The last three derivatives of $[\mathbf{p}\times]$ with respect to the orientation angles of the moving platform are three null matrices.

Let **T1** be the three dimensional matrix $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}\mathbf{A}$ and $\mathbf{T1k}^{\mathbf{j}}$ be the j^{th} column of its k^{th} plane, j, k = 1, ..., 6. The first three planes of **T1** are given by:

$$\left\{ \begin{array}{l} \mathbf{T11}^{i} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cos(\gamma_i) & -\cos(\beta_i) \end{bmatrix}^T \\ \mathbf{T12}^{i} = \begin{bmatrix} 0 & 0 & 0 & -\cos(\gamma_i) & 0 & \cos(\alpha_i) \end{bmatrix}^T \\ \mathbf{T13}^{i} = \begin{bmatrix} 0 & 0 & 0 & \cos(\beta_i) & -\cos(\alpha_i) & 0 \end{bmatrix}^T \end{array} \right\} \tag{35}$$

The last three planes of $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}\mathbf{A}$, i.e. **T14 T15** and **T16**, are 6×6 null matrices. The special form (the first three Plücker coordinates are zero) of **T11**, **T12**, and **T13** shows that the lines of $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}\mathbf{A}$ are lines at infinity (Hunt, 1978).

3.3.2 Formulating the expressions of $\mathbf{B}^{-1}\frac{\partial \mathbf{A}}{\partial \mathbf{x}_i}$

According to Eqs. (17) and (10) we obtain the following expressions for the i^{th} column of $\mathbf{B}^{-1}\frac{\partial \mathbf{A}}{\partial \mathbf{x}_i}$**.** Let **T2** be the three dimensional matrix $\mathbf{B}^{-1}\frac{d\mathbf{A}}{d\mathbf{x}}$. We refer to the k^{th} plane of this matrix, $\mathbf{B}^{-1}\frac{\partial \mathbf{A}}{\partial \mathbf{x}_k}$, by the abbreviation **T2k** where $k = 1\ldots 6$. The expressions of **T21** through **T26** are given in the Appendix-A2.

By substituting the expressions of the i^{th} columns of **T1k** and **T2k**, $k, i = 1\ldots 6$ in Eq. (32) one can see that Eq. (32) is fulfilled. This means that the lines of **T1** and the lines of **T2** intersect each other. This completes the proof that the derivatives of $\mathbf{J}^T$ with respect to position variables are groups of lines. In total, we obtained 36 lines divided to six line-sextuplets with each line-sextuplet representing the derivative of $\mathbf{J}^T$ with respect to one position/orientation variable of the moving platform.

3.4 Simulation results

Numerical and graphical simulations are given below in order to visualize the results. Figure 5 shows the lines of the Jacobian matrix with arrows indicating the direction of the internal forces of the linear actuators. The dotted lines in Fig. 5 are the lines of the derivative of $\mathbf{J}^T$ with respect to the x coordinate of the moving platform.

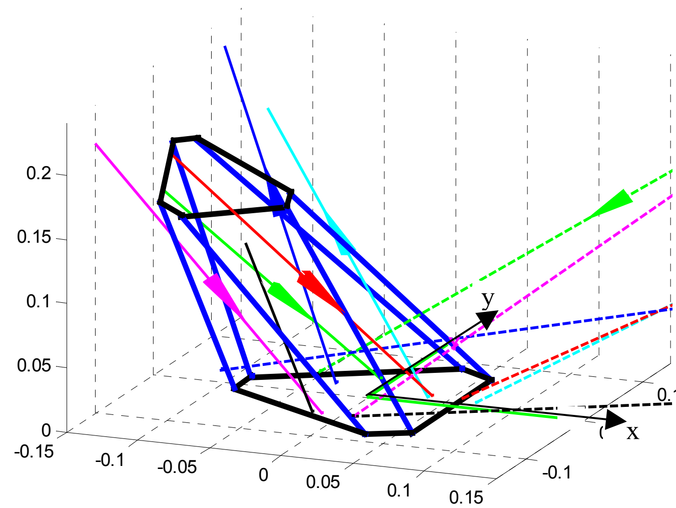


Figure 5: The lines of the Jacobian and the lines of its derivative with respect to x coordinate.

Numerical example:

The following are numerical results of a simulation of the Stewart-Gough 6–6 platform with a moving platform and a base platform having radii of 0.05 and 0.09 m, respectively. The moving platform is positioned at $\mathbf{p} = [-0.1, -0.02, 0.16]^T$ and rotated 30 degrees about the axis [1, 1, 1] relative to the Cartesian coordinate system in Fig. 5. Equations (36) give the transpose of the Jacobian matrix and its derivatives with respect to x, and θ_y, as an example.

$$\mathbf{J}^T = \begin{pmatrix} -0.5742 & -0.6348 & -0.2662 & -0.1886 & -0.6702 & -0.5792 \\ -0.3223 & -0.2715 & -0.0610 & -0.3012 & 0.0799 & 0.3001 \\ 0.7526 & 0.7234 & 0.9620 & 0.9347 & 0.7379 & 0.7579 \\ 0.0154 & 0.0322 & 0.0245 & -0.0441 & -0.0349 & 0.0109 \\ -0.0269 & 0.0070 & 0.0317 & 0.0196 & 0.0107 & -0.0270 \\ 0.0002 & 0.0309 & 0.0088 & -0.0026 & -0.0328 & 0.0190 \end{pmatrix}$$

$$\frac{\partial(\mathbf{J}^T)}{\partial x} = \begin{pmatrix} 3.3431 & 2.4014 & 4.9488 & 5.8132 & 2.7368 & 3.4710 \\ -0.9232 & -0.6932 & -0.0866 & -0.3424 & 0.2661 & 0.9080 \\ 2.1555 & 1.8473 & 1.3640 & 1.0626 & 2.4570 & 2.2932 \\ 0.0440 & 0.0823 & 0.0348 & -0.0501 & -0.1161 & 0.0330 \\ -0.1226 & 0.0976 & 0.1547 & -0.0075 & -0.0213 & -0.1594 \\ -0.1208 & -0.0703 & -0.1163 & 0.2719 & 0.1316 & 0.0131 \end{pmatrix} \quad (36)$$

$$\frac{\partial(\mathbf{J}^T)}{\partial \theta_y} = \begin{pmatrix} -0.1226 & 0.0976 & 0.1547 & -0.0075 & -0.0213 & -0.1594 \\ -0.0433 & 0.0076 & 0.0103 & 0.0355 & -0.0043 & 0.0423 \\ -0.1121 & 0.0885 & 0.0435 & 0.0099 & -0.0189 & -0.1386 \\ -0.0169 & 0.0105 & 0.0032 & 0.0057 & 0.0005 & 0.0135 \\ 0.0373 & -0.0272 & -0.0252 & 0.0011 & 0.0059 & 0.0474 \\ 0.0041 & -0.0092 & -0.0054 & 0.0004 & -0.0019 & -0.0011 \end{pmatrix}$$

It is easy to see, using Eqs. (4) and (32), that the columns of $\mathbf{J}^T$ and its derivatives intersect each other and that the columns of the derivatives of $\mathbf{J}^T$ are a group of lines.

In the remaining part of this paper (sections 4-6) the importance of the derivatives of $\mathbf{J}^T$ is emphasized for active stiffness control (stiffness modulation) in redundant parallel robots. It will be shown that, in particular, this line-based formulation simplifies the analysis of stiffness modulation singularities.

4 APPLICATION OF THE DERIVATIVES OF THE JACOBIAN TO STIFFNESS CONTROL

Stiffness analysis of parallel manipulators plays a key role in determining the degree of adequacy of a given robot to a specific task that involves interaction with the environment. This section relates the Jacobian derivative with active stiffness control, also known as stiffness modulation. The interpretation of this derivative as lines is helpful in determining to what extent the stiffness can be controlled.

4.1 Active stiffness and the derivative of the Jacobian

The stiffness matrix maps the change of the wrench that the robot applies on its environment with the twist deflection of the moving platform. Denoting the i^{th} row of $\mathbf{J}^{T}$ by $\mathbf{J}^{T}{}_{i}$, one can write the elements of the stiffness matrix, **K**, as:

$$k_{ij} = \frac{\partial f_i}{\partial x_j} = \frac{\partial\left(\mathbf{J}^{T}{}_{i}\boldsymbol{\tau}\right)}{\partial x_j} = \frac{\partial \mathbf{J}^{T}{}_{i}}{\partial x_j}\boldsymbol{\tau} + \mathbf{J}^{T}{}_{i}\frac{\partial \boldsymbol{\tau}}{\partial x_j} \tag{37}$$

Unlike the definition in (Gosselin, 1990), this definition includes the stiffness effect of introduced 'preload' (bias forces stemming from, e.g., gravity or external load) in non-redundant manipulators, or antagonistic actuation in redundant robots. This effect is expressed by the term $\frac{\partial \mathbf{J}^{T}{}_{i}}{\partial x_j}\boldsymbol{\tau}$, which is referred to as the 'active stiffness' or 'antagonistic stiffness' (Yi and Freeman, 1993). The other term in Eq. (37) is referred to as the 'passive stiffness' of the manipulator (Yi, et. al., 1992; Kock and Schumacher, 1998). Denoting the j^{th} column of **J** by $\mathbf{J}^{j}$ and treating the m actuators of the robot as springs with a stiffness matrix $\mathbf{K}_d$ in joint space results in:

$$\mathbf{J}^{T}{}_{i}\frac{\partial \boldsymbol{\tau}}{\partial x_j} = \mathbf{J}^{T}{}_{i}\sum_{m}\frac{\partial \boldsymbol{\tau}}{\partial q_m}\frac{\partial q_m}{\partial x_j} = \mathbf{J}^{T}{}_{i}\mathbf{K}_d\mathbf{J}^{j} \tag{38}$$

Stiffness modulation is possible when actuation redundancy is introduced to the system, thus, allowing the use of antagonistic actuation (Cho, et. al., 1989; Yi and Freeman, 1992; kock and Schumacher, 1998; O'brien and Wen, 1999). In this case, the actuation forces are divided into $\boldsymbol{\tau}_p$ and $\boldsymbol{\tau}_h$, where $\boldsymbol{\tau}_p$ denotes the actuation forces balancing the external load and $\boldsymbol{\tau}_h$ denotes the internal actuation forces (antagonistic actuation forces). Antagonistic actuation forces do not affect the net force applied by the moving platform on its environment since they belong to the null space of the Jacobian matrix, Eq. (39).

$$\tau = \tau_p + \tau_h \qquad \mathbf{J}^{T}\tau_p = \mathbf{F}_e \qquad \mathbf{J}^{T}\tau_h = \mathbf{0} \tag{39}$$

Equation (37) can be rewritten in a matrix form as in Eq. (40), where the matrix, $\frac{d\mathbf{J}^T}{dx}$, is a three-dimensional matrix, as in Eq. (11), with the dimensions of 6×m×6 for m actuators (m>6). The multiplication in Eq. (40) should be performed according to Eq. (37), i.e., in order to obtain the active stiffness element, $\mathbf{K1}_{ij,}$ one should take the scalar product of the i^{th} row of the j^{th} plane in the three-dimensional matrix, $\frac{d\mathbf{J}^T}{dx}$, with $\boldsymbol{\tau}$.

$$\mathbf{K} = \frac{\partial \mathbf{J}^T}{\partial x}\tau + \mathbf{J}^T\mathbf{K_d}\mathbf{J} \equiv \mathbf{K1} + \mathbf{K2} \qquad \mathbf{K1} \equiv \frac{\partial \mathbf{J}^T}{\partial x}\tau \qquad \mathbf{K2} \equiv \mathbf{J}^T\mathbf{K_d}\mathbf{J} \tag{40}$$

4.2 stiffness directions and the derivative of the Jacobian

Equation (37) can be written in a matrix form as:

$$\Delta\mathbf{F_e} = \mathbf{K}\Delta\mathbf{x} = \mathbf{K}^1\Delta x_1 + \mathbf{K}^2\Delta x_2 + \mathbf{K}^3\Delta x_3 + \mathbf{K}^4\Delta x_4 + \mathbf{K}^5\Delta x_5 + \mathbf{K}^6\Delta x_6 \tag{41}$$

where $\mathbf{K}^i$ denotes the i'th column of the stiffness matrix, $\mathbf{K}$, $\Delta\mathbf{F}_e$ the change in the reaction wrench of the moving platform on its environment for a positional perturbation $\Delta\mathbf{x}$.

Equation (41) shows that $\mathbf{K}^i$, the i^{th} column of **K,** is the stiffness in the x_i direction since it determines the net change in the moving platform's reaction, $\Delta\mathbf{F}_e$, for a perturbation in the x_i direction. Larger norms of this column cause higher reaction force of the robot.

By Eqs. (40, 41), the i^{th} derivative of the Jacobian matrix maps the joint efforts, $\boldsymbol{\tau}$, into the i^{th} column of the active stiffness matrix, **K1**, thus, modifying the stiffness of the robot in its corresponding direction in the Cartesian world.

Next, the effect of the singularities (rank deficiency) of the Jacobian derivatives on stiffness modulation capabilities is presented.

5 STIFFNESS CONTROL IN REDUNDANT ROBOTS AND SINGULARITY OF THE JACOBIAN DERIVATIVES

Equation (37) gives the expression for the elements of the stiffness matrix. The equation for the j^{th} column of the stiffness matrix is given by:

$$\mathbf{K}^j = \frac{\partial \mathbf{J}^T}{\partial x_j}\boldsymbol{\tau} + \mathbf{J}^T\frac{\partial \boldsymbol{\tau}}{\partial x_j} = \frac{\partial \mathbf{J}^T}{\partial x_j}\boldsymbol{\tau} + \mathbf{J}^T\mathbf{K}_d\mathbf{J}^j \tag{42}$$

The first term of Eq. (42) corresponds to the contribution of the active stiffness gained by redundant actuation. If a given stiffness is required, then the unknowns in Eq. (42) are the actuator forces, τ, needed to cause the required stiffness column $\mathbf{K}^{j}$. The general solution of the static equilibrium problem (Eq. (2)) is given by (Ben-Israel and Greville, 1974):

$$\boldsymbol{\tau} = \mathbf{J}^{T^{+}} \mathbf{F}_{e} + \left(\mathbf{I} - \mathbf{J}^{T^{+}} \mathbf{J}^{T} \right) \xi \tag{43}$$

where the $\mathbf{J}^{T^{+}}$ indicates the Moore-Penrose pseudo inverse of $\mathbf{J}^{T}$ and $\left(\mathbf{I} - \mathbf{J}^{T^{+}} \mathbf{J}^{T}\right)$ is a projector of any arbitrary actuation intensities vector $\xi \in \Re^{m}$ to the kernel of $\mathbf{J}^{T}$. The minimum-norm solution for ξ that satisfies Eq. (42) is given by:

$$\xi = \tilde{\mathbf{J}}^{+} \left[\mathbf{K}^{j} - \mathbf{J}^{T} \mathbf{K}_{d} \mathbf{J}^{j} - \frac{\partial \mathbf{J}^{T}}{\partial x_{j}} \mathbf{J}^{T^{+}} \mathbf{F}_{e} \right] \equiv \tilde{\mathbf{J}}^{+} \mathbf{b} \tag{44}$$

where $\tilde{\mathbf{J}}$ is given by $\tilde{\mathbf{J}} = \dfrac{\partial \mathbf{J}^{T}}{\partial x_{j}} \left(\mathbf{I} - \mathbf{J}^{T^{+}} \mathbf{J}^{T} \right)$.

Equation (44) has an exact (compatible) solution in the general case only if rank ($\tilde{\mathbf{J}}$) = n (where n is the number of the robots' degrees-of-freedom). By the definition $\tilde{\mathbf{J}}$ it is clear that if $\dfrac{\partial \mathbf{J}^{T}}{\partial x_{j}}$ is rank-deficient then in general there is no exact solution to Eq. (42). We note that additional singularities of $\tilde{\mathbf{J}}$ may also stem from the matrix $\left(\mathbf{I} - \mathbf{J}^{T^{+}} \mathbf{J}^{T}\right)$.

Previous works (Yi, Freeman, Tesar, 1989; Yi, Freeman, 1992; Yi, Freeman, Tesar, 1992; Yi and Freeman, 1993) addressed the problem of active stiffness generation via redundancy and mentioned "second order geometric singularities" that prevent exact stiffness modulation. All the above-mentioned works dealt with non-fully parallel manipulators having serial chains supporting the moving platform. The formulations in those works lead to a matrix similar to $\tilde{\mathbf{J}}$ that is composed of an augmented Hessian matrix. The singularities in stiffness modulation were attributed in those works to both the singularity of the Hessian matrix and the singularities of the Jacobian. However, geometrical interpretations were given to the singularity of the Jacobian only. In the above-mentioned investigations the definition the Hessian matrix varied from one work to another. Yi, Freeman, and Tesar, (1989, 1992) defined an augmented Hessian matrix having the Hessians of the inverse kinematics functions of the serial chains, while (Yi and Freeman, 1993) defined this augmented Hessians matrix based on the Hessians of the auxiliary equations that relate the values of passive joints with the values of the active ones. These matrices were not given a geometric interpretation as lines of the Jacobian and its derivatives since the Jacobian matrix of a non-fully parallel manipulator is generally not composed of rows of lines.

The present investigation shows that an arbitrary stiffness modulation is precluded if $\frac{d\mathbf{A}}{d\mathbf{x}}$ or $\mathbf{A}$ are singular. We also obtain, for the first time, a geometric interpretation to the singularity conditions of $\frac{d\mathbf{A}}{d\mathbf{x}}$.

6 GEOMETRIC INTERPRETATION OF THE SINGULARITIES OF $\tilde{\mathbf{J}}$

In this section we will prove that the singularity of $\tilde{\mathbf{J}}$ has a geometric interpretation and is directly related to the linear dependencies of the lines of $\frac{\partial \mathbf{A}}{\partial \mathbf{x}_j}$. The cases where $\mathbf{J}$ is singular (rank($\mathbf{J}$)<n) are excluded since in these cases the robot is singular from structural rigidity considerations. We also limit the discussion to the cases where the number of actuators, m, fulfills $m \geq 2n$ which means that there are enough redundant actuators to fully control a column in the active stiffness matrix, **K1**, of Eq. (40).

proof:

1. From the definition of $\tilde{\mathbf{J}}$ and Eq. (11) one obtains

$$\tilde{\mathbf{J}} = \frac{\partial \mathbf{J}^{T}}{\partial x_j}\left(\mathbf{I} - \mathbf{J}^{T^+}\mathbf{J}^{T} \right) = \left(\frac{\partial \mathbf{B}^{-1}}{\partial x_j}\mathbf{A} + \mathbf{B}^{-1}\frac{\partial \mathbf{A}}{\partial x_j} \right)\left(\mathbf{I} - \mathbf{J}^{T^+}\mathbf{J}^{T} \right) \tag{45}$$

2. By Eq. (9) and the fact that $\mathbf{B}^{-1}$ is a non-singular square matrix one obtains:

$$\mathbf{J}^{T^+}\mathbf{J}^{T} = \left(\mathbf{B}^{-1}\mathbf{A}\right)^{+}\left(\mathbf{B}^{-1}\mathbf{A}\right) = \mathbf{A}^{+}\mathbf{B}^{-1^+}\mathbf{B}^{-1}\mathbf{A} = \mathbf{A}^{+}\mathbf{A} \tag{46}$$

(Note that we used $\left(\mathbf{B}^{-1}\mathbf{A}\right)^{+} = \mathbf{A}^{+}\mathbf{B}^{-1^+}$ which is true only if $\mathbf{B}^{-1}$ and $\mathbf{A}$ are of the same rank, i.e., $\mathbf{A}$ (and $\mathbf{J}$) is non-singular).

3. Applying the properties of the generalized inverse, the first term in Eq. (45) vanishes:

$$\frac{\partial \mathbf{B}^{-1}}{\partial x_j}\mathbf{A}\left(\mathbf{I} - \mathbf{J}^{T^+}\mathbf{J}^{T} \right) = \frac{\partial \mathbf{B}^{-1}}{\partial x_j}\mathbf{A} - \frac{\partial \mathbf{B}^{-1}}{\partial x_j}\mathbf{A}\mathbf{A}^{+}\mathbf{A} = \frac{\partial \mathbf{B}^{-1}}{\partial x_j}\mathbf{A} - \frac{\partial \mathbf{B}^{-1}}{\partial x_j}\mathbf{A} = \mathbf{0} \tag{47}$$

then:

$$\tilde{\mathbf{J}} = \frac{\partial \mathbf{J}^{T}}{\partial x_j}\left(\mathbf{I} - \mathbf{J}^{T^+}\mathbf{J}^{T} \right) = \mathbf{B}^{-1}\frac{\partial \mathbf{A}}{\partial x_j}\left(\mathbf{I} - \mathbf{A}^{+}\mathbf{A} \right) \tag{48}$$

Thus, we proved that the singularity of $\tilde{\mathbf{J}}$ stems from the singularity of $\frac{\partial \mathbf{A}}{\partial \mathbf{x}_j}$.

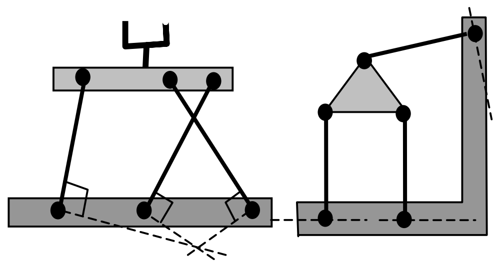
The importance of this proof is that it is easy to visualize the lines of $\frac{\partial \mathbf{A}}{\partial \mathbf{x}_j}$ for planar manipulators and special cases of spatial manipulators. One should recall that the lines of $\frac{\partial \mathbf{A}}{\partial \mathbf{x}_j}$ pass through the joints in the base platform and are perpendicular to the actuators. For planar robots, when more than two lines of $\frac{\partial \mathbf{A}}{\partial \mathbf{x}_j}$ intersect at one point it causes flat pencil singularity of the Jacobian derivatives. Figure 6 shows a flat pencil singularity (Fig. 6-a) and point singularity (Fig. 6-b) of $\frac{\partial \mathbf{A}}{\partial \mathbf{x}_j}$ for a planar 3 DOF non-redundant manipulator. In both configurations the matrix $\tilde{\mathbf{J}}$ has a rank of 2, which means that Eq. (43) has no exact solution for an arbitrary $\mathbf{K}^j$. Figure 7-a and 7-b show flat pencil and point singularities of the matrix **A** (and **J**).

Figure 8 illustrates a redundant planar parallel manipulator with six linear actuators. The dimension of the nullspace of the Jacobian of this planar robot is 3 or higher. This means that we can control the stiffness elements in the j^{th} column of the stiffness matrix provided that rank of the matrix $\tilde{\mathbf{J}}$ associated with this column is no less than 3.

TABLE 1: Singular values of $\tilde{\mathbf{J}}$

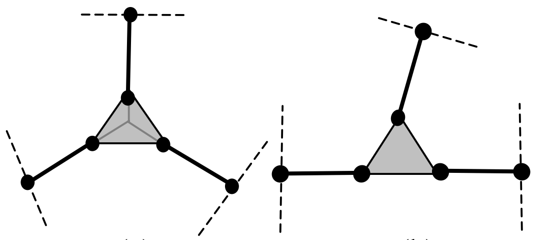
$\tilde{\mathbf{J}}_x$	$\tilde{\mathbf{J}}_y$	$\tilde{\mathbf{J}}_{\theta_z}$
1.2050	1.0353	1.3279
0.6957	0.9204	0.7667
9.3307e-006	4.7369e-016	5.5241e-006
0	0	0
0	0	0
0	0	0

The manipulator in the configuration of Fig. 8 illustrates a singularity of $\tilde{\mathbf{J}}$ (rank($\tilde{\mathbf{J}}$)<3) caused by flat pencil singularity of $\frac{\partial \mathbf{A}}{\partial \mathbf{x}_j}$ since the lines of $\frac{\partial \mathbf{A}}{\partial \mathbf{x}_j}$ intersect in one point as shown in the figure. The singular values of $\tilde{\mathbf{J}}$ are given by Table (1), where the $\tilde{\mathbf{J}}_x$ is associated with the derivative of the Jacobian with respect to the X coordinate, $\tilde{\mathbf{J}}_y$ with respect to the Y coordinate and $\tilde{\mathbf{J}}_{\theta_z}$ with respect to the rotation about the Z axis. The third singular value is small enough to indicate singularity (practically $\tilde{\mathbf{J}}$ has rank 2 since the formulation of $\tilde{\mathbf{J}}$ and the SVD process have cumulative numerical errors and because the dimensions in Fig. 8 were given with 6 digits accuracy).



(a) (b)

Figure 6: Line and flat pencil singularities of the derivatives of the matrix **A.**



(a) (b)

Figure 7: Line and flat pencil singularities of the Jacobian.

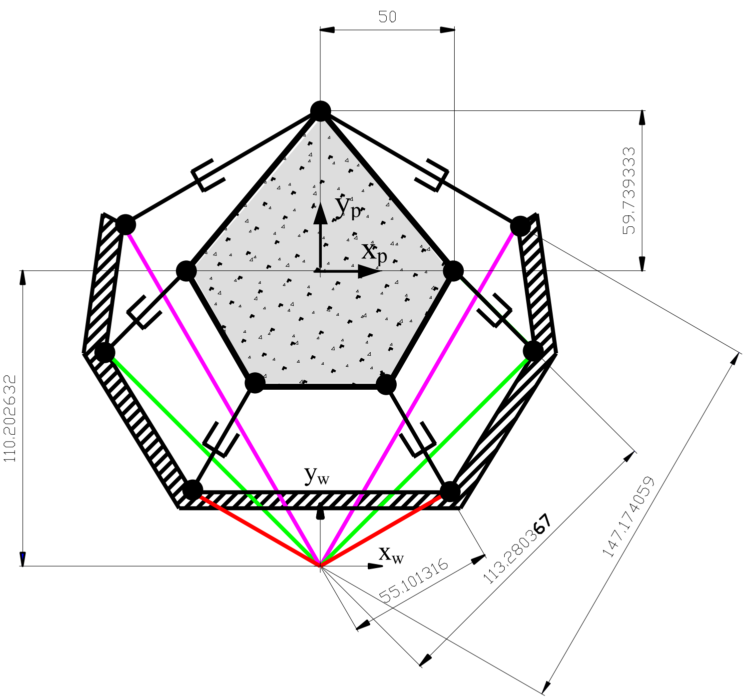


Figure 8: A planar redundant parallel manipulator in a stiffness modulation singularity.

CONCLUSIONS

This paper gave a line-based analytical formulation to the derivatives of the Jacobian of parallel robots. The derivatives were taken with respect to the position/orientation variables of the moving platform rather than time or active joints' variables. The Jacobian derivatives formulation resulted in 36 lines divided into six line-sextuplets, each one representing the derivative of the Jacobian with respect to one position/orientation variable of the moving platform.

The problem of controlling the stiffness of the robot in Cartesian space (also known as the stiffness modulation problem) was solved and each derivative of the Jacobian was associated with active stiffness modification in a corresponding direction in space. The significance of the line-based formulation of the Jacobian derivatives for stiffness modulation was emphasized and used to interpret stiffness modulation singularities of redundant parallel robots. It was shown that these stiffness modulation singularities are function of **A** (the instantaneous direct kinematics matrix) and its derivative. This interpretation allows the use of line geometry tools for stiffness modulation singularity analysis similarly to the line-based structural rigidity singularity analysis of parallel robots. In this sense, this paper adds to the knowledge of previous investigations that analyze the stiffness modulation singularities stemming only from the classical first-order singularities of the Jacobian matrix.

The authors believe that the line geometry-based formulation of the Jacobian derivatives facilitates the geometrical interpretation of rigidity, stability and dynamics that are based on the derivative of the Jacobian matrix.

APPENDIX A-1

The i^{th} rows of the Jacobians $\mathbf{J}_\alpha$, $\mathbf{J}_\beta$, and $\mathbf{J}_\gamma$ are given by:

$$
\left\{
\begin{aligned}
\left[\mathbf{J}_\alpha\right]_i = \Bigg[& -\frac{s_{\alpha_i}}{q_i}, \frac{c_{\alpha_i} c_{\beta_i}}{q_i s_{\alpha_i}}, \frac{c_{\alpha_i} c_{\gamma_i}}{q_i s_{\alpha_i}}, \frac{c_{\alpha_i} p_z c_{\beta_i} - c_{\alpha_i} p_y c_{\gamma_i} + c_{\alpha_i} b_{i_y} c_{\gamma_i} - c_{\alpha_i} b_{i_z} c_{\beta_i}}{q_i s_{\alpha_i}}, \frac{-q_i c_{\gamma_i} + p_z s_{\alpha_i}^2 - b_{i_z} s_{\alpha_i}^2 + c_{\alpha_i} p_x c_{\gamma_i} - c_{\alpha_i} b_{i_x} c_{\gamma_i}}{q_i s_{\alpha_i}}, \\
& \frac{q_i c_{\beta_i} - p_y s_{\alpha_i}^2 + b_{i_y} s_{\alpha_i}^2 - c_{\alpha_i} p_x c_{\beta_i} + c_{\alpha_i} b_{i_x} c_{\beta_i}}{q_i s_{\alpha_i}} \Bigg] \\
\left[\mathbf{J}_\beta\right]_i = \Bigg[& \frac{c_{\alpha_i} c_{\beta_i}}{q_i s_{\beta_i}}, -\frac{s_{\beta_i}}{q_i}, \frac{c_{\beta_i} c_{\gamma_i}}{q_i s_{\beta_i}}, \frac{q_i c_{\gamma_i} - p_z s_{\beta_i}^2 + b_{i_z} s_{\beta_i}^2 - c_{\beta_i} p_y c_{\gamma_i} + c_{\beta_i} b_{i_y} c_{\gamma_i}}{q_i s_{\beta_i}}, \frac{-c_{\alpha_i} p_z c_{\beta_i} + c_{\beta_i} p_x c_{\gamma_i} + c_{\alpha_i} b_{i_z} c_{\beta_i} - c_{\beta_i} b_{i_x} c_{\gamma_i}}{q_i s_{\beta_i}}, \\
& \frac{-q_i c_{\alpha_i} + p_x s_{\beta_i}^2 - b_{i_x} s_{\beta_i}^2 + c_{\beta_i} p_y c_{\alpha_i} - c_{\beta_i} b_{i_y} c_{\alpha_i}}{q_i s_{\beta_i}} \Bigg] \\
\left[\mathbf{J}_\gamma\right]_i = \Bigg[& \frac{c_{\alpha_i} c_{\gamma_i}}{q_i s_{\gamma_i}}, \frac{c_{\beta_i} c_{\gamma_i}}{q_i s_{\gamma_i}}, -\frac{s_{\gamma_i}}{q_i}, \frac{-q_i c_{\beta_i} + p_y s_{\gamma_i}^2 - b_{i_y} s_{\gamma_i}^2 + c_{\gamma_i} p_z c_{\beta_i} - c_{\gamma_i} b_{i_z} c_{\beta_i}}{q_i s_{\gamma_i}}, \frac{q_i c_{\alpha_i} - p_x s_{\gamma_i}^2 + b_{i_x} s_{\gamma_i}^2 - c_{\gamma_i} p_z c_{\alpha_i} + c_{\gamma_i} b_{i_z} c_{\alpha_i}}{q_i s_{\gamma_i}}, \\
& \frac{c_{\alpha_i} p_y c_{\gamma_i} - c_{\beta_i} p_x c_{\gamma_i} + c_{\beta_i} b_{i_x} c_{\gamma_i} - c_{\alpha_i} b_{i_y} c_{\gamma_i}}{q_i s_{\gamma_i}} \Bigg]
\end{aligned}
\right\}
$$

APPENDIX A-2

The following equations give the explicit expression of the i^{th} column of **T2k**, k,i=1, ..., 6.

$$
\mathrm{T21} = \left[\frac{s_{\alpha_i}^2}{q_i}, -\frac{c_{\alpha_i} c_{\beta_i}}{q_i}, -\frac{c_{\alpha_i} c_{\gamma_i}}{q_i}, -\frac{p_z c_{\alpha_i} c_{\beta_i}}{q_i} + \frac{p_y c_{\alpha_i} c_{\gamma_i}}{q_i} + \frac{b_{i_z} c_{\alpha_i} c_{\beta_i}}{q_i} - \frac{b_{i_y} c_{\alpha_i} c_{\gamma_i}}{q_i}, -\frac{p_z s_{\alpha_i}^2}{q_i} - \frac{p_x c_{\alpha_i} c_{\gamma_i}}{q_i} + \frac{b_{i_z} s_{\alpha_i}^2}{q_i} + \frac{b_{i_x} c_{\alpha_i} c_{\gamma_i}}{q_i}, \frac{p_y s_{\alpha_i}^2}{q_i} + \frac{p_x c_{\alpha_i} c_{\beta_i}}{q_i} - \frac{b_{i_y} s_{\alpha_i}^2}{q_i} - \frac{b_{i_x} c_{\alpha_i} c_{\beta_i}}{q_i} \right]
$$

$$
\mathrm{T22} = \left[-\frac{c_{\alpha_i} c_{\beta_i}}{q_i}, \frac{s_{\beta_i}^2}{q_i}, -\frac{c_{\beta_i} c_{\gamma_i}}{q_i}, \frac{p_z s_{\beta_i}^2}{q_i} + \frac{p_y c_{\beta_i} c_{\gamma_i}}{q_i} - \frac{b_{i_z} s_{\beta_i}^2}{q_i} - \frac{b_{i_y} c_{\beta_i} c_{\gamma_i}}{q_i}, \frac{p_z c_{\alpha_i} c_{\beta_i}}{q_i} - \frac{p_x c_{\beta_i} c_{\gamma_i}}{q_i} - \frac{b_{i_z} c_{\alpha_i} c_{\beta_i}}{q_i} + \frac{b_{i_x} c_{\beta_i} c_{\gamma_i}}{q_i}, -\frac{p_y c_{\alpha_i} c_{\beta_i}}{q_i} - \frac{p_x s_{\beta_i}^2}{q_i} + \frac{b_{i_y} c_{\alpha_i} c_{\beta_i}}{q_i} + \frac{b_{i_x} s_{\beta_i}^2}{q_i} \right]
$$

$$
\mathrm{T23} = \left[-\frac{c_{\alpha_i} c_{\gamma_i}}{q_i}, -\frac{c_{\beta_i} c_{\gamma_i}}{q_i}, \frac{s_{\gamma_i}^2}{q_i}, -\frac{p_z c_{\beta_i} c_{\gamma_i}}{q_i} - \frac{p_y s_{\gamma_i}^2}{q_i} + \frac{b_{i_z} c_{\beta_i} c_{\gamma_i}}{q_i} + \frac{b_{i_y} s_{\gamma_i}^2}{q_i}, \frac{p_z c_{\alpha_i} c_{\gamma_i}}{q_i} + \frac{p_x s_{\gamma_i}^2}{q_i} - \frac{b_{i_z} c_{\alpha_i} c_{\gamma_i}}{q_i} - \frac{b_{i_x} s_{\gamma_i}^2}{q_i}, -\frac{p_y c_{\alpha_i} c_{\gamma_i}}{q_i} + \frac{p_x c_{\beta_i} c_{\gamma_i}}{q_i} + \frac{b_{i_y} c_{\alpha_i} c_{\gamma_i}}{q_i} - \frac{b_{i_x} c_{\beta_i} c_{\gamma_i}}{q_i} \right]
$$

$$
\begin{aligned}
T24 = \Bigg[& -\frac{c_{\alpha_i} p_z c_{\beta_i} - c_{\alpha_i} p_y c_{\gamma_i} + c_{\alpha_i} b_{i_y} c_{\gamma_i} - c_{\alpha_i} b_{i_z} c_{\beta_i}}{q_i}, -\frac{-c_{\beta_i} p_y c_{\gamma_i} + c_{\beta_i} b_{i_y} c_{\gamma_i} + q_i c_{\gamma_i} - p_z s_{\beta_i}^2 + b_{i_z} s_{\beta_i}^2}{q_i}, -\frac{-c_{\gamma_i} b_{i_z} c_{\beta_i} + c_{\gamma_i} p_z c_{\beta_i} - q_i c_{\beta_i} + p_y s_{\gamma_i}^2 - b_{i_y} s_{\gamma_i}^2}{q_i}, \\
& -\frac{p_z \left(-c_{\beta_i} p_y c_{\gamma_i} + c_{\beta_i} b_{i_y} c_{\gamma_i} + q_i c_{\gamma_i} - p_z s_{\beta_i}^2 + b_{i_z} s_{\beta_i}^2\right)}{q_i} + \frac{p_y \left(-c_{\gamma_i} b_{i_z} c_{\beta_i} + c_{\gamma_i} p_z c_{\beta_i} - q_i c_{\beta_i} + p_y s_{\gamma_i}^2 - b_{i_y} s_{\gamma_i}^2\right)}{q_i} + \frac{b_{i_z} \left(-c_{\beta_i} p_y c_{\gamma_i} + c_{\beta_i} b_{i_y} c_{\gamma_i} + q_i c_{\gamma_i} - p_z s_{\beta_i}^2 + b_{i_z} s_{\beta_i}^2\right)}{q_i} \\
& -\frac{b_{i_y} \left(-c_{\gamma_i} b_{i_z} c_{\beta_i} + c_{\gamma_i} p_z c_{\beta_i} - q_i c_{\beta_i} + p_y s_{\gamma_i}^2 - b_{i_y} s_{\gamma_i}^2\right)}{q_i}, \frac{p_z (c_{\alpha_i} p_z c_{\beta_i} - c_{\alpha_i} p_y c_{\gamma_i} + c_{\alpha_i} b_{i_y} c_{\gamma_i} - c_{\alpha_i} b_{i_z} c_{\beta_i})}{q_i} - \frac{p_x \left(-c_{\gamma_i} b_{i_z} c_{\beta_i} + c_{\gamma_i} p_z c_{\beta_i} - q_i c_{\beta_i} + p_y s_{\gamma_i}^2 - b_{i_y} s_{\gamma_i}^2\right)}{q_i} \\
& -\frac{b_{i_z} (c_{\alpha_i} p_z c_{\beta_i} - c_{\alpha_i} p_y c_{\gamma_i} + c_{\alpha_i} b_{i_y} c_{\gamma_i} - c_{\alpha_i} b_{i_z} c_{\beta_i})}{q_i} + \frac{b_{i_x} \left(-c_{\gamma_i} b_{i_z} c_{\beta_i} + c_{\gamma_i} p_z c_{\beta_i} - q_i c_{\beta_i} + p_y s_{\gamma_i}^2 - b_{i_y} s_{\gamma_i}^2\right)}{q_i}, -\frac{p_y (c_{\alpha_i} p_z c_{\beta_i} - c_{\alpha_i} p_y c_{\gamma_i} + c_{\alpha_i} b_{i_y} c_{\gamma_i} - c_{\alpha_i} b_{i_z} c_{\beta_i})}{q_i} \\
& +\frac{p_x \left(-c_{\beta_i} p_y c_{\gamma_i} + c_{\beta_i} b_{i_y} c_{\gamma_i} + q_i c_{\gamma_i} - p_z s_{\beta_i}^2 + b_{i_z} s_{\beta_i}^2\right)}{q_i} + \frac{b_{i_y} (c_{\alpha_i} p_z c_{\beta_i} - c_{\alpha_i} p_y c_{\gamma_i} + c_{\alpha_i} b_{i_y} c_{\gamma_i} - c_{\alpha_i} b_{i_z} c_{\beta_i})}{q_i} - \frac{b_{i_x} \left(-c_{\beta_i} p_y c_{\gamma_i} + c_{\beta_i} b_{i_y} c_{\gamma_i} + q_i c_{\gamma_i} - p_z s_{\beta_i}^2 + b_{i_z} s_{\beta_i}^2\right)}{q_i} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
T25 = \Bigg[& -\frac{c_{\alpha_i} p_x c_{\gamma_i} - c_{\alpha_i} b_{i_x} c_{\gamma_i} - q_i c_{\gamma_i} + p_z s_{\alpha_i}^2 - b_{i_z} s_{\alpha_i}^2}{q_i}, -\frac{-c_{\alpha_i} p_z c_{\beta_i} + c_{\beta_i} p_x c_{\gamma_i} + c_{\alpha_i} b_{i_z} c_{\beta_i} - c_{\beta_i} b_{i_x} c_{\gamma_i}}{q_i}, -\frac{c_{\gamma_i} b_{i_z} c_{\alpha_i} - c_{\gamma_i} p_z c_{\alpha_i} + q_i c_{\alpha_i} - p_x s_{\gamma_i}^2 + b_{i_x} s_{\gamma_i}^2}{q_i}, \\
& -\frac{p_z (-c_{\alpha_i} p_z c_{\beta_i} + c_{\beta_i} p_x c_{\gamma_i} + c_{\alpha_i} b_{i_z} c_{\beta_i} - c_{\beta_i} b_{i_x} c_{\gamma_i})}{q_i} + \frac{p_y \left(c_{\gamma_i} b_{i_z} c_{\alpha_i} - c_{\gamma_i} p_z c_{\alpha_i} + q_i c_{\alpha_i} - p_x s_{\gamma_i}^2 + b_{i_x} s_{\gamma_i}^2\right)}{q_i} + \frac{b_{i_z} (-c_{\alpha_i} p_z c_{\beta_i} + c_{\beta_i} p_x c_{\gamma_i} + c_{\alpha_i} b_{i_z} c_{\beta_i} - c_{\beta_i} b_{i_x} c_{\gamma_i})}{q_i} \\
& -\frac{b_{i_y} \left(c_{\gamma_i} b_{i_z} c_{\alpha_i} - c_{\gamma_i} p_z c_{\alpha_i} + q_i c_{\alpha_i} - p_x s_{\gamma_i}^2 + b_{i_x} s_{\gamma_i}^2\right)}{q_i}, \frac{p_z \left(c_{\alpha_i} p_x c_{\gamma_i} - c_{\alpha_i} b_{i_x} c_{\gamma_i} - q_i c_{\gamma_i} + p_z s_{\alpha_i}^2 - b_{i_z} s_{\alpha_i}^2\right)}{q_i} - \frac{p_x \left(c_{\gamma_i} b_{i_z} c_{\alpha_i} - c_{\gamma_i} p_z c_{\alpha_i} + q_i c_{\alpha_i} - p_x s_{\gamma_i}^2 + b_{i_x} s_{\gamma_i}^2\right)}{q_i} \\
& -\frac{b_{i_z} \left(c_{\alpha_i} p_x c_{\gamma_i} - c_{\alpha_i} b_{i_x} c_{\gamma_i} - q_i c_{\gamma_i} + p_z s_{\alpha_i}^2 - b_{i_z} s_{\alpha_i}^2\right)}{q_i} + \frac{b_{i_x} \left(c_{\gamma_i} b_{i_z} c_{\alpha_i} - c_{\gamma_i} p_z c_{\alpha_i} + q_i c_{\alpha_i} - p_x s_{\gamma_i}^2 + b_{i_x} s_{\gamma_i}^2\right)}{q_i}, -\frac{p_y \left(c_{\alpha_i} p_x c_{\gamma_i} - c_{\alpha_i} b_{i_x} c_{\gamma_i} - q_i c_{\gamma_i} + p_z s_{\alpha_i}^2 - b_{i_z} s_{\alpha_i}^2\right)}{q_i} \\
& +\frac{p_x (-c_{\alpha_i} p_z c_{\beta_i} + c_{\beta_i} p_x c_{\gamma_i} + c_{\alpha_i} b_{i_z} c_{\beta_i} - c_{\beta_i} b_{i_x} c_{\gamma_i})}{q_i} + \frac{b_{i_y} \left(c_{\alpha_i} p_x c_{\gamma_i} - c_{\alpha_i} b_{i_x} c_{\gamma_i} - q_i c_{\gamma_i} + p_z s_{\alpha_i}^2 - b_{i_z} s_{\alpha_i}^2\right)}{q_i} - \frac{b_{i_x} (-c_{\alpha_i} p_z c_{\beta_i} + c_{\beta_i} p_x c_{\gamma_i} + c_{\alpha_i} b_{i_z} c_{\beta_i} - c_{\beta_i} b_{i_x} c_{\gamma_i})}{q_i} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
T26 = \Bigg[& -\frac{-c_{\alpha_i} p_x c_{\beta_i} + c_{\alpha_i} b_{i_x} c_{\beta_i} + q_i c_{\beta_i} - p_y s_{\alpha_i}^2 + b_{i_y} s_{\alpha_i}^2}{q_i}, -\frac{-c_{\beta_i} b_{i_y} c_{\alpha_i} + c_{\beta_i} p_y c_{\alpha_i} - q_i c_{\alpha_i} + p_x s_{\beta_i}^2 - b_{i_x} s_{\beta_i}^2}{q_i}, -\frac{c_{\alpha_i} p_y c_{\gamma_i} - c_{\beta_i} p_x c_{\gamma_i} + c_{\beta_i} b_{i_x} c_{\gamma_i} - c_{\alpha_i} b_{i_y} c_{\gamma_i}}{q_i}, \\
& -\frac{p_z \left(-c_{\beta_i} b_{i_y} c_{\alpha_i} + c_{\beta_i} p_y c_{\alpha_i} - q_i c_{\alpha_i} + p_x s_{\beta_i}^2 - b_{i_x} s_{\beta_i}^2\right)}{q_i} + \frac{p_y (c_{\alpha_i} p_y c_{\gamma_i} - c_{\beta_i} p_x c_{\gamma_i} + c_{\beta_i} b_{i_x} c_{\gamma_i} - c_{\alpha_i} b_{i_y} c_{\gamma_i})}{q_i} + \frac{b_{i_z} \left(-c_{\beta_i} b_{i_y} c_{\alpha_i} + c_{\beta_i} p_y c_{\alpha_i} - q_i c_{\alpha_i} + p_x s_{\beta_i}^2 - b_{i_x} s_{\beta_i}^2\right)}{q_i} \\
& -\frac{b_{i_y} (c_{\alpha_i} p_y c_{\gamma_i} - c_{\beta_i} p_x c_{\gamma_i} + c_{\beta_i} b_{i_x} c_{\gamma_i} - c_{\alpha_i} b_{i_y} c_{\gamma_i})}{q_i}, \frac{p_z \left(-c_{\alpha_i} p_x c_{\beta_i} + c_{\alpha_i} b_{i_x} c_{\beta_i} + q_i c_{\beta_i} - p_y s_{\alpha_i}^2 + b_{i_y} s_{\alpha_i}^2\right)}{q_i} - \frac{p_x (c_{\alpha_i} p_y c_{\gamma_i} - c_{\beta_i} p_x c_{\gamma_i} + c_{\beta_i} b_{i_x} c_{\gamma_i} - c_{\alpha_i} b_{i_y} c_{\gamma_i})}{q_i} \\
& -\frac{b_{i_z} \left(-c_{\alpha_i} p_x c_{\beta_i} + c_{\alpha_i} b_{i_x} c_{\beta_i} + q_i c_{\beta_i} - p_y s_{\alpha_i}^2 + b_{i_y} s_{\alpha_i}^2\right)}{q_i} + \frac{b_{i_x} (c_{\alpha_i} p_y c_{\gamma_i} - c_{\beta_i} p_x c_{\gamma_i} + c_{\beta_i} b_{i_x} c_{\gamma_i} - c_{\alpha_i} b_{i_y} c_{\gamma_i})}{q_i}, -\frac{p_y \left(-c_{\alpha_i} p_x c_{\beta_i} + c_{\alpha_i} b_{i_x} c_{\beta_i} + q_i c_{\beta_i} - p_y s_{\alpha_i}^2 + b_{i_y} s_{\alpha_i}^2\right)}{q_i} \\
& +\frac{p_x \left(-c_{\beta_i} b_{i_y} c_{\alpha_i} + c_{\beta_i} p_y c_{\alpha_i} - q_i c_{\alpha_i} + p_x s_{\beta_i}^2 - b_{i_x} s_{\beta_i}^2\right)}{q_i} + \frac{b_{i_y} \left(-c_{\alpha_i} p_x c_{\beta_i} + c_{\alpha_i} b_{i_x} c_{\beta_i} + q_i c_{\beta_i} - p_y s_{\alpha_i}^2 + b_{i_y} s_{\alpha_i}^2\right)}{q_i} - \frac{b_{i_x} \left(-c_{\beta_i} b_{i_y} c_{\alpha_i} + c_{\beta_i} p_y c_{\alpha_i} - q_i c_{\alpha_i} + p_x s_{\beta_i}^2 - b_{i_x} s_{\beta_i}^2\right)}{q_i} \Bigg]
\end{aligned}
$$

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