ON GRASSMANN'S PRODUCTS AND CLIFFORD'S DUAL UNIT

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Introduction

The history of Vector Algebra is traced back to the end of the Eighteenth century. We consider the complex number as the origin of vector algebra since complex numbers are two ordered pair where their addition describes the parallelogram rule and their product describes planar rotation. This feature was well known in the beginning of the nineteenth century and there were numerous attempts to extend this idea into three dimensions or, in Hamilton's phrase, to “multiply triplets".

For years scientists searched for the proper solution, which, considering early nineteenth century point of view, was not easy to produce. As we know, several basic concepts are not preserved in modern vector algebra - commutativity, associativity and "division" rule.

In 1843 Hamilton invented the quaternions – four elements which their algebra leads to our scalar, and vector multiplication of vectors and also allows three-dimensional rotation of vectors. It should be mentioned that even though only the commutative rule is not preserved in quaternion algebra it was still difficult to accept by mathematician of those times.

Prior to Hamilton, in 1840, two investigators arrived at similar results, obviously not through quaternion algebra. Rodriguez developed the three-dimensional vector rotation formula in a general form (unlike Hamilton who developed the expression for particular case of rotating a vector about a perpendicular quaternion axis [Altmann, 1999]). Grassmann found of what we know as the scalar and vector multiplication of vectors. These are classical cases in which revolutionary discoveries went unnoticed because the investigators were not known before, and they publish only very few works in this field. Grassmann was a mathematics teacher at technical high school and could not reach a university position. Rodriguez was a banker and his famous 60 pages long paper is the only one he published.

Grassmann wrote a book on the calculus of extensives which nobody fully understood . It was sent to Gauss, Hamilton Saint-Venant and other but letters from that period show that no one actually went beyond the first pages of the first version of his book.

The interesting part of Grassmann's calculus is that it deals with geometric entities, such as: points, lines, areas. etc. and defines multiplication rules for these entities which result in a higher order entity, in a coordinate free language. The two main multiplication rules suggested by Grassmann are the inner and the outer product which are the scalar product and a similar to vector cross product of our days (he suggested other multiplication rules as well).

In his theory Grassmann treated differently, as different entities, what we call today *'free vectors'* and *'line vectors'*. Clifford adopted the same idea in 1873, where he made the distinction between *vector* and a *roto*r by inventing the dual unit.

In the sequel, Grassmann's and Clifford's approaches to the combination of the two different forms of vectors, are compared.

Product of extensives

Hermann Gunther Grassmann (1809-1877) lived in Stettin Germany where he taught mathematics at various schools "despite his constant attempts to attain a university post" [Crowe, 1967]. In 1840 he wrote an essay on "Theorie der Ebbe Und Flut" [1840] which contains first ideas on geometrical sum and difference and geometrical product of lines. In his forward to his main work Ausdehnungslehre [1844], were he elaborated on his ideas of adding line not along one direction, on the associative rules of his algebra, and the anti-commutative rule of vector multiplication, Grassmann wrote: " Thus I feel entitled to hope that I have found in this new analysis the only natural method according to which mathematics should be applied to nature .. Thus I decided to make the presentation, extension and application of this analysis a task of my life.“

Grassmann treated geometrical and physical entities called "extensives" which can be point, lines, oriented circles, tetrahedron, etc.,

Extensives are divided into levels (steps) and can be added and multiplied. However, addition can take place only in the same level while multiplication resulted in a higher level. We will concentrate on the *'outer product'* rule which is one of several rules defined by Grassmann.

We will define the extensives need for the purpose of this paper.

- First extensive is a point.
- A sum of points is another (weighted) point while the difference of points is a vector given by:

$$a - b = u \tag{1}$$

where a and b are points and *u* is a vector.

Grassmann showed that if a vector is defined as a difference between two points than the parallelogram low of vector addition is preserved.

Grassmann defined his basic *"outer product"* of two extensives, a and b, and a scalar, k, as follows [Forder, 1960]:

1. [a k b] = k [a b]
2. [k a b] = k [a b]
3. [a (b + c)] = [a b] + [a c]
4. [(b + c) a)] = [b a] + [c a]
5. [a a] = 0

from which one obtains the anti-commutative rule (not easily accepted by those days mathematicians)

$$[a\ b] = -\ [b\ a] \tag{2}$$

When a and b are points (extensive of "step one") than the outer multiplication [a b] represent a vector anchored to a line. This is a basic concept which distinguish between a *vector* which is the difference of two points and can be moved parallel to itself without changing its properties and a *line vector* which is a result of outer multiplication of two points and is bounded to a line.

- Outer product of three points [a b c] results in a directed area with the magnitude of twice the triangle formed by points a, b and c.

Based on the above assumptions, outer product of a point, a, and a vector (b - c) is:

$$[a\ (b - c)\ ] = [a\ b] + [a\ d] \tag{3}$$

where [a d] = - [a c].

The result of an outer product of a point and a vector is therefore, a line vector of a length and direction of the vector (b - c) passing through point a.

- Outer product of two vectors (a - b) and (p - q) is:

$$[(a -b) (p - q)] = [a (p - q)] - [b (p - q)] \tag{4}$$

According to the previous result these are two parallel and equal line vectors (p - q) through two different points - a and b. This entity is defined as a *bi-vector* and it is, as the line vector, of "step two". It is easy to see that the physical interpretation of the bi-vector is a pure moment and hence a free vector according to our days interpretation (but not according to Grassmann's since a vector which is a difference between two points is an extensive of "step one").

It should be mentioned that the word "bi-vector" is used here since in Grassmann's theory there are tri-vector as well. However it has a different meaning than Clifford's definition of the word "bi" in his theory on bi-quaternions.

- Sum of line-vectors passing through the same point is a line-vector. Assume n line-vectors passing through point a:

$[a\ p_1] + [a\ p_2] +[a\ p_3] +\ldots+ [a\ p_n] = [a(p_1 - a)] + [a(a\ p_2 - a)] + [a(p_3 - a)] + \ldots.+[a(p_n - a)]$

since [a a]=0 by definition 5.

Using the distributive law, 3, one obtains:

$$[a(p_1 - a)] + [a(p_2 - a)] + [a(p_3 - a)] + \ldots.+[a(p_n - a)] = [a((p_1 + p_2 + p_3 +\ldots p_n) - na)]. \tag{5}$$

Now since the sum of points is a point it implies that $(p_1 + p_2 + p_3 +\ldots p_n) – na$ is a vector and hence when outer multiplied by point a results in a line-vector through a.

However, when a general case is considered namely the line-vectors do not necessarily pass through the same point, then the result is a line-vector and a bi-vector. Take two points p_1 and p_2 their product $[p_1 p_2]$ yields a line-vector.

$$[p_1\ p_2] = [p_1\ (p_2 - p_1)] = [p_1\ (p_2 - p_1)] + [a\ (p_2 - p_1)] - [a\ (p_2 - p_1)] = [a\ (p_2 - p_1)] = [p_1\ (p_2 - p_1) – a\ (p_2 - p_1)] = [a\ (p_2 - p_1)] + [(p_1 - a)\ (p_2 - p_1)] \tag{6}$$

Hence, a rotor $[p_1\ p_2]$ is equal to a rotor through another point a and a bi-vector (parallel to $(p_2 - p_1)$ and through p_1 and a).

When any number of line-vectors are considered, then they can always be treated as a line-vectors through a common point, which is a line-vector, and sum of bi-vectors, which is a also bi-vector.

It is interesting to examine the outer product of line-vectors and bi-vectors.

- Outer product of two line-vectors

Line-vectors are the result of outer multiplication of two points. Hence multiplication of two line vectors are equivalent to outer product of four points.

- Outer product of four points a, b, c, and d is six times the volume of the tetrahedron abcd. It is a scalar and can not be interpreted as simply cross product or dot product of modern vector algebra.

- Outer product of a line-vector and a bi-vector

The outer product of a line-vector and a bi-vector can be obtained by an outer multiplication of three vectors since a bi-vector is the result of an outer product of two vectors. This is defined as a tri-vector

- Outer product of two bi-vectors

The outer product of two bi-vectors is composed of a product of four vectors since a bi-vector is a product of two vectors (see above). Hence it is a product of a vector and a tri-vector which is zero [Forder, 1965].

- Outer product of a unit tri-vector (a tri-vector with a magnitude equal to 1), "annihilates a vector or a bi-vector , reduce a rotor to its vector , and a leaf to its bi-vector" [Forder, pp.108].

We will examine these results in light of Clifford's dual unit.

Clifford's dual unit

William Kingdon Clifford (1845-1879) took another approach to the combination of *motors* which are sum of *rotors* and *vectors*. He become acquainted with Grassmann's work only few years later as he mentioned in his paper of 1878 [Clifford, 1878]. In his work similar to Grassmann he distinguish between *vectors* which are entities having a magnitude and direction (free-vectors) and *rotors* which are entities having magnitude direction and position (line-vector). Sum of vectors is a vector while sum of rotors is generally not a rotor but a combination of a rotor and a vector namely a *motor* [Clifford, 1873].

Clifford Became acquainted and impressed by Grassmann's work only after he published his 1873 paper on bi-quaternions. Grassmann's work influenced Clifford in his latter works where he developed Clifford algebra which real, complex, and quaternion multiplication’s rules in n dimensional space are particular cases. It is interesting to note that the entire book on Clifford Algebra published in 1998 [Baylis, 1996] emphasizes its basic assumption that a square of a vector is equal to the square of its magnitude, but the dual unit appears only in a very short note on bi-quaternions.

We will go briefly over Clifford derivation of the dual unit as a necessary unit in combination of motors.

Clifford's goal was to determine the operation that converts one motor into another which he concluded to be a *bi-quaternion*. To follow his derivation, we rewrite, for convenience, some of his definitions.

A *vector* is a quantity with magnitude and direction (e.g. linear velocity or moment).

A *rotor* is a quantity with magnitude, direction, and position (e.g. rotational velocity about a fixed axis or force along line of action).

A *motor* is the sum of two or more rotors, which can be represented as a wrench or twist about a certain screw. For example, the sum of arbitrary system of forces is, in general, not a force but a combination of force and moment.

We start by repeating the basic arguments in Clifford's paper (See [Shoham, 1999] for details).

Analogous to Hamilton's "tensor-versor " operation, which converts one vector into the other, Clifford defined the "tensor-twist" for non-intersecting rotors. Since a rotor has a specific direction and position, a "tensor-twist" - a screw motion about the common perpendicular and stretching (or compressing) along the resulting rotor axis - is the operation which converts one rotor into the other.

Mathematically, this is given by:

$$T\,\mathbf{b} = \mathbf{a} \tag{7}$$

where $\mathbf{a}$ and $\mathbf{b}$ are rotors and T is the "tensor-twist" operation.

"Tensor - twist" is also the operation which relates two equal-pitched motors (a motor can be considered as a rotor associated with a pitch).

Since motors are the sum of two or more rotors, it can be written as:

$$\mathbf{A} = \mathrm{m}\,\mathbf{a} + \mathrm{n}\,\mathbf{b} \qquad (8)$$

where m and n are scalars and **a** and **b** are rotors.

Assuming that the "tensor-twist" operator in a linear one, the following equation holds:

$$T(\mathrm{m}\,\mathbf{a} + \mathrm{n}\,\mathbf{b}) = \mathrm{m}\,\mathbf{c} + \mathrm{n}\,\mathbf{d} \qquad (9)$$

where $T\ \mathbf{a} = \mathbf{c}$ and $T\ \mathbf{b} = \mathbf{d}$.

To arrive at his goal, namely, to find the ratio of two motors Clifford "divided" a motor **B** by a motor **A**.

Let **B**' be a motor with rotor part equal to motor **B** and with the same pitch as motor **A**, then:

$$\mathbf{B} = \mathbf{B'} + \beta \qquad (10)$$

where β is a vector parallel to the axis of **B** (adding vector β to **B'** changes its pitch and maintain its direction and position).

Dividing by **A** while keeping in mind that **B'** and **A** have the same pitch one obtains:

$$\mathbf{B} \,/\, \mathbf{A} = T + \beta \,/\, \mathbf{A} \qquad (11)$$

Observing the rightmost term, Clifford realized that one needs to apply a special unit which when applied to a motor converts it to a vector parallel and proportional to its rotor part. This unit is known as the dual number unit, ε, with the special property:

$$\varepsilon^2 = 0\,, \qquad (12)$$

reflecting the fact that when it is applied twice to a motor, the result is zero.

Grassmann Vs. Clifford Definitions

Comparing Grassmann and Clifford's definitions for ‘combination of vectors’ one arrives at the following results:

According to Grassmann the sum of line-vectors is a combination of a line-vector and a bi-vector. In Clifford's words sum of a rotors is generaly a motor (which is a combination of a line-vector and a free-vector).

If we assume, as Clifford suggested, that a free vector carry the dual unit ε while a line-vector (rotor) does not, then we would expect an outer multiplication of two vectors to vanish. This is however not the case in Grassmann’s representation since the outer multiplication of two vectors is a bi-vector . Moreover, vector according to Grassmann has a different "step" than a line-vector or a bi-vector. A vector is of "step" one while a line-vector or bi-vector is of "step" two and hence can not be added.

We have to conclude that Clifford's vector (which carries the dual unit) is actually a bi-vector in Grassmann's terms. This can be shown from the above results.

Grassmann's bi-vector is not fixed to a line (and can physically interpreted as a moment) since its definition is a difference between two identical and opposite line-vectors through two different points. Hence this is a free vector in Clifford's definition and has the dual nature.

Outer multiplication of a bi-vector with a (Grassmann's) vector results in a tri-vector which, according to Clifford, can be viewed as still being multiplied by a dual unit. Once this is further multiplied by a vector the result is zero, which can be interpreted in Clifford's words as multiplication of two free vectors each carries the dual unit.

Also, in Grassmann's derivation outer product of a unit tri-vector reduce a line-vector to a free vector. This is similar to the phrase appeared in Clifford's 1873 paper "when the dual unit is applied to any motor, it changes it into a vector (namely, free-vector) parallel to its axis and proportional to the rotor part of it". When it is applied twice to a motor the result is zero.

Conclusions

Clifford and Grassmann arrived at similar results from two totally different approaches. Both realized the difference between free-vector and a line vector and looked for a specific operation which distinguish between the two.

It seems that our free vector (used for example, to describe pure moments) is defined in Grassmann's terms as bi-vector, it is of "step" two and obtained as a result of outer product of two vectors of "step" one (which is basically the difference between two points and should not be confused with our use of vectors in physics). Line–vectors are obtained as an outer product (not difference) of two points and it is also of "step" two.

Clifford distinguished between the two by the dual unit ε, which accompany free-vectors. The fact that application of twice the dual unit to a motor is zero means in Grassmann' word that outer product of two bi-vectors is zero. Hence it might be concluded based on the above that Grassmann's bi-vector is equivalent to Clifford's free-vector.

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