Remarks on “Hidden” Lines in Parallel Robots

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Abstract

Derivatives of the Jacobian matrix of robot manipulators are used for rigidity, stability, dynamics and compliant motion analysis. This paper investigates the properties of the derivatives of the Jacobian matrices of fully-parallel manipulators from a geometrical point of view. These derivatives are taken with respect to the moving platform’s position/orientation coordinates rather than time or active joints’ coordinates. The paper presents a special formulation of the Jacobian matrix that simplifies the sought derivatives and their geometric interpretation. Similarly to the Jacobian matrix, its derivatives are proven to represent also a group of lines and the geometrical interrelations between these two groups of lines are presented.

1. Introduction

Line geometry has been applied by several researchers to the kinematics and statics of parallel manipulators [Merlet, 1989; Colling and Long, 1995; Ben Horin, 1997; Simaan, 1999; Pottman, Peternell, and Ravani, 1999]. Line geometry is used because the rows of the Jacobian matrix in a linearly actuated fully-parallel manipulator are the Plücker line coordinates of the axes of its extensible links [Hunt, Samuel, and McAree, 1991]. Hence, linear dependence of these lines determines the conditions for instability and singularity of a parallel manipulator as Dandurand has shown in the context of stability of spatial grids [Dandurand, 1984].

The present paper analyses the derivatives of the Jacobian matrix with respect to the six position variables of the moving platform and seeks their geometrical interpretation. The derivative of the Jacobian matrix is important in rigidity analysis [Byung, Freeman, and Tesar 1989; Kock and Schumacher, 1998], dynamic manipulability analysis [Yoshikawa, 1990], and force-controlled compliant motions [Dutré, Bruyninckx, and Schutter, 1997].

In contrast to the numerous investigations devoted to the formulation of parallel manipulators’ Jacobian matrix e.g., [Cleary and Uebel, 1994; Simaan, Glozman, and Shoham, 1998; Tsai, 1998], there

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are only a few studies addressing the formulation of its derivative. Dutré, et al., [1997] addressed this problem and obtained a closed form analytic expression of the inverse Jacobian matrix derivative with respect to time and with respect to the joint active variables. Merlet and Gosselin [1991] formulated the time derivative of the Jacobian of a fully manipulator for use in acceleration analysis.

The present work investigates the geometric interpretation of the derivatives of the direct Jacobian matrix with respect to the position/orientation variables of the moving platform, and evaluates its contribution to the manipulator's rigidity.

To the best of the authors' knowledge, there are no prior studies that show that the derivative of a parallel manipulator's Jacobian matrix has a geometric interpretation as a separate group of lines – a fact that can be further used for rigidity and compliant motion analysis.

2.0 Jacobian matrix formulation

Consider a general Stewart-Gough type parallel manipulator subject to a wrench $\mathbf{F}_{env} = [\mathbf{f}_{env}^t, \mathbf{m}_{env}^t]^t$ applied by the environment, Fig. 1. Let $\dot{\mathbf{x}}$ denote the end effector twist and $\dot{\mathbf{q}}$ the corresponding active joints' rates. The commonly used expression of the Jacobian matrix is:

$$\dot{\mathbf{q}} = \mathbf{J}\dot{\mathbf{x}}, \tag{1}$$

which is the inverse of that of serial manipulators' $\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$.

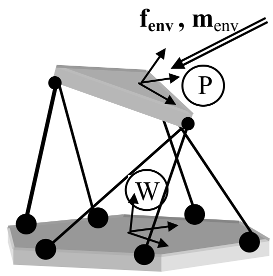

Figure 1: Typical Stewart-Gough Manipulator

In this paper we use Eq. (1) to map the end effector twist, $\dot{\mathbf{x}}$, to active joint rates, $\dot{\mathbf{q}}$. The Jacobian matrix is also used to relate the required active joints' forces for a desired external wrench $\mathbf{F}_e = [\mathbf{f}_e^t, \mathbf{m}_e^t]^t$ to be exerted on the environment ($\mathbf{F}_e = -\mathbf{F}_{env}$).

$$\mathbf{J}^t\boldsymbol{\tau} = \mathbf{F}_e \tag{2}$$

Using the loop closure method [Ma and Angeles, 1992], or the static equilibrium method [Cleary and Uebel, 1994; Simaan, Glozman, and Shoham, 1998; Simaan, 1999], along with Eqs. (1) and (2), respectively, yields the commonly used formulation of the Jacobian matrix.

$$\mathbf{J} = \begin{bmatrix} \hat{\mathbf{l}}_1 & {}^w\mathbf{R}_p\mathbf{u}_1 \times \hat{\mathbf{l}}_1 \\ \vdots & \vdots \\ \hat{\mathbf{l}}_6 & {}^w\mathbf{R}_p\mathbf{u}_6 \times \hat{\mathbf{l}}_6 \end{bmatrix} \tag{3}$$

where $\hat{\mathbf{l}}_i$ denotes a unit vector of the *i*th active prismatic joint pointing from its spherical joint at the base to its spherical joint at the moving platform. We denote the platform-attached and the base-attached coordinate systems by the letters P and W, respectively (Fig. 1). Accordingly, ${}^w\mathbf{R}_p$ is the rotation matrix transforming vectors from P to W, and $\mathbf{u}_i$ is the position vector of the *i*th spherical joint in P.

In order to interpret the Jacobian matrix as lines, the following basic definitions of line geometry are reviewed. A given sextuplet of numbers $[l_{vx}, l_{vy}, l_{vz}, l_{mx}, l_{my}, l_{mz}]$ represents a line in space only when it belongs to a five-dimensional quadratic manifold called the Grassmannian [Merlet, 1989; Pellegrini, 1997], the Plücker hypersurface [Graustein 1930, Sommerville, 1934] or Klein quadric [Pottman, Peternell, and Ravani, 1999; Pellegrini, 1997] or in other words it fulfils Eq. (5).

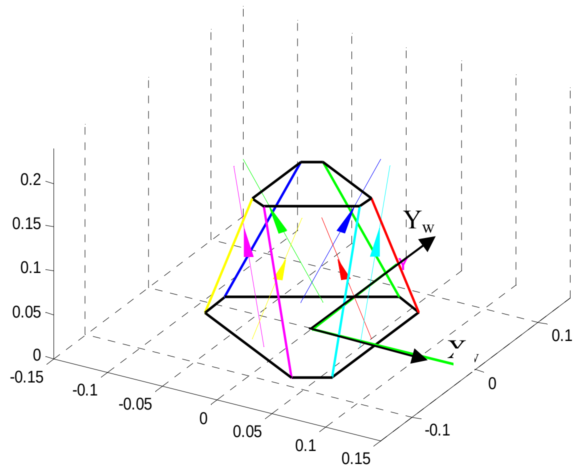

Figure 3: Lines of the Jacobian in W

$$l_{vx}\, l_{mx} + l_{vy}\, l_{my} + l_{vz}\, l_{mz} = 0 \tag{5}$$

Observing Eq. (3), it is clear that the rows of the Jacobian are the Plücker ray coordinates of lines along the prismatic actuators. This physical interpretation is correct in a coordinate system having its origin located at the center of the moving platform. In this representation each row of the Jacobian matrix is a function of ${}^{W}\mathbf{R}_p\mathbf{u}_i$ and the direction numbers of $\hat{\mathbf{l}}_i$, which are both functions of the moving platform position.

3. Interpretation of the Jacobian matrix's lines in the stationary versus the moving platform coordinate system

Consider another representation of the Jacobian matrix in the form:

$$\mathbf{J}_b{}^t\tau = \mathbf{F}_b \tag{6}$$

where $\mathbf{F}_b=[\mathbf{f}_b{}^t,\ \mathbf{m}_b{}^t]^t$ represents the wrench exerted by the base rather than the moving platform on the environment (see Fig 2). By using simple statics equations and representing $\mathbf{F}_b$ by $\mathbf{F}_e$ one obtains:

$$\mathbf{A}\tau = \mathbf{B}\mathbf{F}_e \tag{7}$$

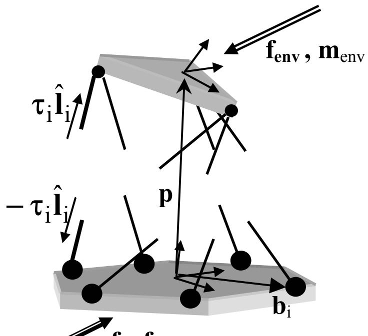

Figure 2: Static equilibrium on

where: $$\mathbf{A} = \begin{bmatrix} \hat{\mathbf{l}}_1 & \cdots & \hat{\mathbf{l}}_6 \\ \mathbf{b}_1\times\hat{\mathbf{l}}_1 & \cdots & \mathbf{b}_6\times\hat{\mathbf{l}}_6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ [\mathbf{p}\times] & \mathbf{I} \end{bmatrix} \tag{8}$$

I – 3×3 unit matrix

b_i – position vector of the spherical joint of the ith prismatic actuator at the base in W coordinate system.

[p×] – skew-symmetric matrix representing vector multiplication.

$$[\mathbf{p}\times]=\begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix} \quad (9)$$

Eqs. (6) and (7) yield:

$$\mathbf{J}^t = \mathbf{B}^{-1}\mathbf{A} \quad (10)$$

Where $\mathbf{B}^{-1}$ is given by:

$$\mathbf{B}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ [-\mathbf{p}\times] & \mathbf{I} \end{bmatrix} \quad (11)$$

Contrary to ${}^{w}\mathbf{R}_p\mathbf{u}_i$, which is a varying vector in W, the vector $\mathbf{b}_i$ is constant in W. This simplifies the expression of the derivative of $\mathbf{J}^t$. It should be mentioned that the change suggested above is not a change of coordinate system from tool to world coordinate system, which clearly does not affect the derivation, but it is a change of the point about which the moments of the lines are calculated. In this formulation the lines of **A** are fixed in W and therefore their derivative is easily shown to be lines as will be shown later.

The physical interpretation of multiplying a Plücker line's coordinates by the matrix $\mathbf{B}^{-1}$ is a translation the line while maintaining its direction. Figure 3 shows a 6–6 Stewart-Gough platform manipulator with the lines of the Jacobian in W. Another important feature of $\mathbf{B}^{-1}$ is that its determinant is equal to 1, which means that the above multiplication does not add to the singularities of **J**.

4.0 Formulation of the derivative of the Jacobian matrix

The derivatives of $\mathbf{J}^t$ with respect to the moving platform's position variables is obtained from Eq. (10) as:

$$\frac{d\mathbf{J}^t}{d\mathbf{x}} = \frac{d\mathbf{B}^{-1}}{d\mathbf{x}}\mathbf{A} + \mathbf{B}^{-1}\frac{d\mathbf{A}}{d\mathbf{x}} \quad (12)$$

The matrices $\frac{d\mathbf{J}^t}{d\mathbf{x}}$, $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}$, $\frac{d\mathbf{A}}{d\mathbf{x}}$ are three-dimensional 6×6×6 matrices for six degrees-of-freedom manipulators. The *i*th plane of these matrices is their derivative with respect to the *i*th position/orientation coordinate, x_i, of the moving platform.

The multiplication in Eq. 12 is performed plane by plane, i.e., for obtaining the derivative of $\mathbf{J}^t$ with respect to the *i*th position/orientation variable one should multiply the *i*th plane of $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}$ with **A** and multiply $\mathbf{B}^{-1}$ with the *i*th plane of $\frac{d\mathbf{A}}{d\mathbf{x}}$.

The derivative of $\mathbf{B}^{-1}$ is simple and yields a matrix whose structure is similar to $\mathbf{B}^{-1}$ so the first expression on the right hand side of Eq. (12) yields a matrix whose columns are the translated lines of **A** under the transformation $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}$. If the derivative $\frac{d\mathbf{A}}{d\mathbf{x}}$ yields a matrix whose columns are also lines and the translated lines $\mathbf{B}^{-1}\frac{d\mathbf{A}}{d\mathbf{x}}$ intersect the lines of $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}\mathbf{A}$, then the derivative of **J** is also a matrix with lines as its columns. This is true since any linear combination of two given intersecting lines spans a flat pencil of lines [Graustein, 1930].

4.1 derivative of the matrix A

The matrix **A** in Eq. (8) is composed of the lines along the robot's prismatic joints. Each unit vector along these lines is characterized by its direction cosines α_i, β_i, and γ_i:

$$\hat{\mathbf{l}}_i = [\cos(\alpha_i), \cos(\beta_i), \cos(\gamma_i)]^t \tag{13}$$

The matrix $\frac{d\mathbf{A}}{d\mathbf{x}}$ is a three-dimensional 6×6×6 matrix with the *i*th plane being the derivative of **A** with respect to the *i*th position/orientation coordinate of the moving platform, $\frac{\partial \mathbf{A}}{\partial \mathbf{x}_i}$. Since **A** has the lines $\mathbf{l}_i$ as its columns we are interested in finding the derivatives of these lines.

Using Eq. (8) while keeping in mind that the vectors $\mathbf{b}_i$ are constant one can write:

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where

$$\frac{\partial \mathbf{l}_j}{\partial x_i} = \frac{\partial \mathbf{l}_j}{\partial \alpha_j}\frac{\partial \alpha_j}{\partial x_i} + \frac{\partial \mathbf{l}_j}{\partial \beta_j}\frac{\partial \beta_j}{\partial x_i} + \frac{\partial \mathbf{l}_j}{\partial \gamma_j}\frac{\partial \gamma_j}{\partial x_i} \tag{15}$$

Define $\mathbf{J}_\alpha, \mathbf{J}_\beta, \mathbf{J}_\gamma \in \Re^{n\times n}$ such that:

$$J_{\alpha_{m,n}} = \frac{\partial \alpha_m}{\partial x_n},\ J_{\beta_{m,n}} = \frac{\partial \beta_m}{\partial x_n},\ J_{\gamma_{m,n}} = \frac{\partial \gamma_m}{\partial x_n} \tag{16}$$

In order to write Eq. (15) in a matrix form, we define three matrices $\frac{d\mathbf{A}}{d\alpha}$, $\frac{d\mathbf{A}}{d\beta}$, and $\frac{d\mathbf{A}}{d\gamma}$:

$$\frac{\partial \mathbf{A}}{\partial \alpha} = \begin{bmatrix} \frac{\partial \mathbf{l}_1}{\partial \alpha_1} & \cdots & \frac{\partial \mathbf{l}_n}{\partial \alpha_n} \end{bmatrix} \quad \frac{\partial \mathbf{A}}{\partial \beta} = \begin{bmatrix} \frac{\partial \mathbf{l}_1}{\partial \beta_1} & \cdots & \frac{\partial \mathbf{l}_n}{\partial \beta_n} \end{bmatrix} \quad \frac{\partial \mathbf{A}}{\partial \gamma} = \begin{bmatrix} \frac{\partial \mathbf{l}_1}{\partial \gamma_1} & \cdots & \frac{\partial \mathbf{l}_n}{\partial \gamma_n} \end{bmatrix} \tag{17}$$

We also define $\mathbf{J}_{d\alpha_i}, \mathbf{J}_{d\beta i}, \mathbf{J}_{d\gamma i} \in \Re^{n\times n}$ as three diagonal matrices having on their main diagonals the *i*th columns of $\mathbf{J}_\alpha$, $\mathbf{J}_\beta$, and $\mathbf{J}_\gamma$ respectively.

Using these definitions one can write Eq. (14) in matrix form as:

$$\frac{\partial \mathbf{A}}{\partial x_i} = \frac{\partial \mathbf{A}}{\partial \alpha}\mathbf{J}_{d_{\alpha i}} + \frac{\partial \mathbf{A}}{\partial \beta}\mathbf{J}_{d_{\beta i}} + \frac{\partial \mathbf{A}}{\partial \gamma}\mathbf{J}_{d_{\gamma i}} \tag{18}$$

The derivatives of the lines with respect to their variables (keeping in mind that $\mathbf{b}_i$ is constant) are:

$$\frac{\partial \mathbf{l}_i}{\partial \alpha_i} = \left[\left[-\sin(\alpha_i) \quad 0 \quad 0\right], \mathbf{b}_i \times \left[-\sin(\alpha_i) \quad 0 \quad 0\right]\right] \tag{19}$$

$$\frac{\partial \mathbf{l}_i}{\partial \beta_i} = \left[\left[0 \quad -\sin(\beta_i) \quad 0\right], \mathbf{b}_i \times \left[0 \quad -\sin(\beta_i) \quad 0\right]\right] \tag{20}$$

$$\frac{\partial \mathbf{l}_i}{\partial \gamma_i} = \left[\left[0 \quad 0 \quad -\sin(\gamma_i)\right], \mathbf{b}_i \times \left[0 \quad 0 \quad -\sin(\gamma_i)\right]\right] \tag{21}$$

It can be seen that Eqs. (19-21) are also lines that intersect the lines of the matrix $\mathbf{A}$ at points $\mathbf{b}_i$.

Since only two independent variables are required to define the direction of a line in 3D the following constraint equation exists:

$$\cos(\alpha_i)^2 + \cos(\beta_i)^2 + \cos(\gamma_i)^2 = 1 \tag{22}$$

Differentiating Eq. (22) with respect to x_i and solving for $\frac{\partial \gamma_i}{\partial x_i}$ yields:

$$\frac{\partial \gamma_i}{\partial x_i} = \frac{-c_{\alpha_i} s_{\alpha_i}}{c_{\gamma_i} s_{\gamma_i}} \frac{\partial \alpha_i}{\partial x_i} + \frac{-c_{\beta_i} s_{\beta_i}}{c_{\gamma_i} s_{\gamma_i}} \frac{\partial \beta_i}{\partial x_i} \tag{23}$$

Where the abbreviations s_α and c_α stand for $\sin(\alpha)$ and $\cos(\alpha)$ respectively.

Substituting Eq. (23) in (15) yields:

$$\frac{\partial \mathbf{l}_i}{\partial x_j} = \begin{bmatrix} -s_{\alpha i} \\ 0 \\ c_{\alpha_i} s_{\alpha_i} / c_{\gamma_i} \\ \mathbf{b}_i \times \begin{bmatrix} -s_\alpha \\ 0 \\ c_{\alpha_i} s_{\alpha_i} / c_{\gamma_i} \end{bmatrix} \end{bmatrix} \frac{\partial \alpha_i}{\partial x_j} + \begin{bmatrix} 0 \\ -s_{\beta_i} \\ c_{\beta_i} s_{\beta_i} / c_{\gamma_i} \\ \mathbf{b}_i \times \begin{bmatrix} 0 \\ -s_{\beta_i} \\ c_{\beta_i} s_{\beta_i} / c_{\gamma_i} \end{bmatrix} \end{bmatrix} \frac{\partial \beta_i}{\partial x_j} \tag{24}$$

The first and the second brackets in Eq. (24) are $\frac{\partial \mathbf{l}_i}{\partial \alpha_i}$ and $\frac{\partial \mathbf{l}_i}{\partial \beta_i}$, respectively. Both these brackets represent lines according to Eq. (5) and it is easy to see that both are perpendicular to $\mathbf{l}_i$. The expressions

$\frac{\partial \alpha_i}{\partial x_j}$ and $\frac{\partial \beta_i}{\partial x_j}$ are scalars. Consequently, the columns of $\frac{\partial \mathbf{A}}{\partial x_i}$ in Eq. (14) are lines that pass through the spherical joints in the points $\mathbf{b}_i$ and belong to the flat pencils of $\frac{\partial \mathbf{l}_i}{\partial \alpha_i}$ and $\frac{\partial \mathbf{l}_i}{\partial \beta_i}$.

Summarizing this section, we conclude that the lines of the derivative of **A** are perpendicular to the lines of **A** and intersect them in the points $\mathbf{b}_i$, i.e., in the spherical joints at the base platform. We use this fact next to show that the derivative of the Jacobian matrix is also a group of lines.

4.2 Deriving $\mathbf{J}_\alpha$, $\mathbf{J}_\beta$, and $\mathbf{J}_\gamma$

Equations (18) and (16) give the expression for the derivative of **A** as a function of three Jacobian matrices $\mathbf{J}_\alpha$, $\mathbf{J}_\beta$, and $\mathbf{J}_\gamma$. This section derives the expressions of these Jacobians. Figure 4 depicts a fully parallel robot with six independent closed loops. Each loop is governed by the loop equation:

$$\mathbf{p} + {}^W\mathbf{R}_p\mathbf{u}_i = \mathbf{b}_i + q_i\hat{\mathbf{l}}_i \tag{25}$$

Taking the time derivative of Eq. (25) yields:

$$\dot{\mathbf{p}} - {}^W\mathbf{R}_p\mathbf{u}_i \times {}^W\boldsymbol{\omega}^p = \dot{q}_i\hat{\mathbf{l}}_i + q_i\dot{\hat{\mathbf{l}}}_i \tag{26}$$

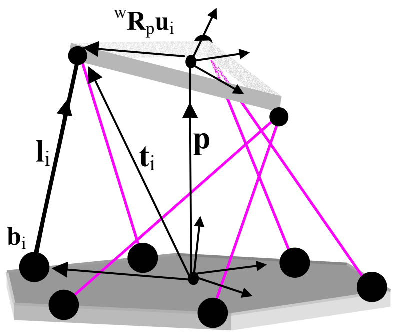

Figure 4: Kinematic closed loops

where q_i represents the length of the *i*th prismatic joint, **p** the position of the moving platform in W, and ${}^W\boldsymbol{\omega}^p$ the angular velocity of the moving platform in W. Rewriting the right-hand side of Eq. (26) in terms of the vector of linear/angular velocities of the moving platform, $\dot{\mathbf{x}} = \left[\dot{\mathbf{p}}^t \quad ({}^W\boldsymbol{\omega}^p)^t\right]^t$, yields:

$$\dot{\mathbf{p}} - {}^W\mathbf{R}_p\mathbf{u}_i \times {}^W\boldsymbol{\omega}^p = \left[\mathbf{I} \quad , \left[-({}^W\mathbf{R}_p\mathbf{u}_i)\times\right]\right]\dot{\mathbf{x}} \equiv \mathbf{M}_i\dot{\mathbf{x}} \tag{27}$$

Expression $\dot{q}_i\hat{\mathbf{l}}_i$ in Eq. (26) is expressed in terms of $\dot{\mathbf{x}}$ by using the velocity relation $\dot{\mathbf{q}} = \mathbf{J}\dot{\mathbf{x}}$ with reference to the *i*th row of **J** as $\mathbf{J}_i$, and using Eq. (13) for $\hat{\mathbf{l}}_i$:

$$\dot{q}_i\hat{\mathbf{l}}_i = \begin{bmatrix} \cos(\alpha_i)\mathbf{J}_i \\ \cos(\beta_i)\mathbf{J}_i \\ \cos(\gamma_i)\mathbf{J}_i \end{bmatrix}_{3\times 6} \dot{\mathbf{x}} \equiv \mathbf{N}_i\dot{\mathbf{x}} \tag{28}$$

Substituting back into Eq. (26) yields:

$$q_i\begin{bmatrix} -\sin(\alpha_i)\dot{\alpha}_i \\ -\sin(\beta_i)\dot{\beta}_i \\ -\sin(\gamma_i)\dot{\gamma}_i \end{bmatrix} = [\mathbf{M}_i - \mathbf{N}_i]\ \dot{\mathbf{x}} \tag{29}$$

Solving Eq. (29) for its unknowns $\dot{\alpha}_i, \dot{\beta}_i$, and $\dot{\gamma}_i$ yields:

$$\dot{\alpha}_i = \left[\frac{-1}{q_i \sin(\alpha_i)}[\mathbf{M}_i - \mathbf{N}_i]_1\right]\dot{\mathbf{x}}\ ,\ \dot{\beta}_i = \left[\frac{-1}{q_i \sin(\beta_i)}[\mathbf{M}_i - \mathbf{N}_i]_2\right]\dot{\mathbf{x}}\ ,\ \dot{\gamma}_i = \left[\frac{-1}{q_i \sin(\gamma_i)}[\mathbf{M}_i - \mathbf{N}_i]_3\right]\dot{\mathbf{x}} \quad (30)$$

Where $[\mathbf{M}_i\text{-}\mathbf{N}_i]_j$ is the *j*th row of $[\mathbf{M}_i\text{-}\mathbf{N}_i]$, j = 1, 2, 3. Equation (31) gives the *i*th rows of $\mathbf{J}_\alpha$, $\mathbf{J}_\beta$, and $\mathbf{J}_\gamma$ as:

$$[\mathbf{J}_\alpha]_i = \left[\frac{-1}{q_i \sin(\alpha_i)}[\mathbf{M}_i - \mathbf{N}_i]_1\right],\ [\mathbf{J}_\beta]_i = \left[\frac{-1}{q_i \sin(\beta_i)}[\mathbf{M}_i - \mathbf{N}_i]_2\right],\ [\mathbf{J}_\gamma]_i = \left[\frac{-1}{q_i \sin(\gamma_i)}[\mathbf{M}_i - \mathbf{N}_i]_3\right] \quad (31)$$

This completes the formulation of the necessary terms in Eq. (18) and, thus, the derivative of **A** is fully defined and proven to be a matrix whose columns are lines. These lines are perpendicular to the lines of **A** and intersect them at the spherical joints at the base, $\mathbf{b}_i$. What remains is to show that the sum of the terms in Eq. (12) gives a set of lines.

4.3 Intersection of the lines of $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}_i}\mathbf{A}$ and the lines of $\mathbf{B}^{-1}\frac{d\mathbf{A}}{d\mathbf{x}_i}$

Observing Eq. (12), one concludes that the last three planes of $\frac{d\mathbf{J}}{dx}$, i.e. $\frac{\partial \mathbf{J}}{\partial x_k}$ k=4,5,6, are the translated lines of $\frac{d\mathbf{A}}{d\mathbf{x}}$ under the transformation $\mathbf{B}^{-1}$.

This can be written as

$$\frac{\partial \mathbf{J}^t}{\partial \mathbf{x}_i} = \mathbf{B}^{-1}\frac{\partial \mathbf{A}}{\partial \mathbf{x}_i} \qquad i=4,5,6. \quad (32)$$

It remains to prove that the $\frac{\partial \mathbf{J}}{\partial x_i}$ for i = 1, 2, 3 represent lines. In order to prove this, we must prove that the lines of $\frac{\partial \mathbf{B}^{-1}}{\partial \mathbf{x}_i}\mathbf{A}$ intersect the lines of $\mathbf{B}^{-1}\frac{\partial \mathbf{A}}{\partial \mathbf{x}_i}$.

The following proof relies on the condition of intersection between two given lines, $\mathbf{l} = [l_1, l_2, l_3, l_4, l_5, l_6]^t$ and $\mathbf{m} = [m_1, m_2, m_3, m_4, m_5, m_6]^t$. This condition is given in Eq. (33) and has the interpretation of the moment of a force acting along line **l** about line **m** [Hunt, 1978].

$$l_1m_4 + l_2m_5 + l_3m_6 + l_4m_1 + l_5m_2 + l_6m_1 = 0 \quad (33)$$

This is proven symbolically using Maple® (a symbolic manipulation program) and also verified numerically with a numerical and a graphical simulation using Matlab®.

The i^{th} column of **A** and the i^{th} row of **J** are given by:

$$\begin{array}{l} \mathbf{J}_i = [c_{\alpha_i}, c_{\beta_i}, c_{\gamma_i}, p_z c_{\beta_i} - p_y c_{\gamma_i} + b_{i_y} c_{\gamma_i} - b_{i_z} c_{\beta_i}, -p_z c_{\alpha_i} + p_x c_{\gamma_i} + b_{i_z} c_{\alpha_i} - b_{i_x} c_{\gamma_i}, p_y c_{\alpha_i} - p_x c_{\beta_i} + b_{i_x} c_{\beta_i} - b_{i_y} c_{\alpha_i}] \\ \mathbf{A}^i = [c_{\alpha_i}, c_{\beta_i}, c_{\gamma_i}, b_{i_y} c_{\gamma_i} - b_{i_z} c_{\beta_i}, b_{i_z} c_{\alpha_i} - b_{i_x} c_{\gamma_i}, b_{i_x} c_{\beta_i} - b_{i_y} c_{\alpha_i}] \end{array} \tag{34}$$

The i^{th} rows of $\mathbf{J}_\alpha$, $\mathbf{J}_\beta$, and $\mathbf{J}_\gamma$ are given by Eq. (35)

$$\left\{ \begin{array}{l} [\mathbf{J}_\alpha]_i = \left[-\frac{s_{\alpha_i}}{q_i}, \frac{c_{\alpha_i} c_{\beta_i}}{q_i s_{\alpha_i}}, \frac{c_{\alpha_i} c_{\gamma_i}}{q_i s_{\alpha_i}}, \frac{c_{\alpha_i} p_z c_{\beta_i} - c_{\alpha_i} p_y c_{\gamma_i} + c_{\alpha_i} b_{i_y} c_{\gamma_i} - c_{\alpha_i} b_{i_z} c_{\beta_i}}{q_i s_{\alpha_i}}, \frac{-q_i c_{\gamma_i} + p_z s_{\alpha_i}^2 - b_{i_z} s_{\alpha_i}^2 + c_{\alpha_i} p_x c_{\gamma_i} - c_{\alpha_i} b_{i_x} c_{\gamma_i}}{q_i s_{\alpha_i}}, \right. \\ \qquad \left. \frac{q_i c_{\beta_i} - p_y s_{\alpha_i}^2 + b_{i_y} s_{\alpha_i}^2 - c_{\alpha_i} p_x c_{\beta_i} + c_{\alpha_i} b_{i_x} c_{\beta_i}}{q_i s_{\alpha_i}} \right] \\ [\mathbf{J}_\beta]_i = \left[\frac{c_{\alpha_i} c_{\beta_i}}{q_i s_{\beta_i}}, -\frac{s_{\beta_i}}{q_i}, \frac{c_{\beta_i} c_{\gamma_i}}{q_i s_{\beta_i}}, \frac{q_i c_{\gamma_i} - p_z s_{\beta_i}^2 + b_{i_z} s_{\beta_i}^2 - c_{\beta_i} p_y c_{\gamma_i} + c_{\beta_i} b_{i_y} c_{\gamma_i}}{q_i s_{\beta_i}}, \frac{-c_{\alpha_i} p_z c_{\beta_i} + c_{\beta_i} p_x c_{\gamma_i} + c_{\alpha_i} b_{i_z} c_{\beta_i} - c_{\beta_i} b_{i_x} c_{\gamma_i}}{q_i s_{\beta_i}}, \right. \\ \qquad \left. \frac{-q_i c_{\alpha_i} + p_x s_{\beta_i}^2 - b_{i_x} s_{\beta_i}^2 + c_{\beta_i} p_y c_{\alpha_i} - c_{\beta_i} b_{i_y} c_{\alpha_i}}{q_i s_{\beta_i}} \right] \\ [\mathbf{J}_\gamma]_i = \left[\frac{c_{\alpha_i} c_{\gamma_i}}{q_i s_{\gamma_i}}, \frac{c_{\beta_i} c_{\gamma_i}}{q_i s_{\gamma_i}}, -\frac{s_{\gamma_i}}{q_i}, \frac{-q_i c_{\beta_i} + p_y s_{\gamma_i}^2 - b_{i_y} s_{\gamma_i}^2 + c_{\gamma_i} p_z c_{\beta_i} - c_{\gamma_i} b_{i_z} c_{\beta_i}}{q_i s_{\gamma_i}}, \frac{q_i c_{\alpha_i} - p_x s_{\gamma_i}^2 + b_{i_x} s_{\gamma_i}^2 - c_{\gamma_i} p_z c_{\alpha_i} + c_{\gamma_i} b_{i_z} c_{\alpha_i}}{q_i s_{\gamma_i}}, \right. \\ \qquad \left. \frac{c_{\alpha_i} p_y c_{\gamma_i} - c_{\beta_i} p_x c_{\gamma_i} + c_{\beta_i} b_{i_x} c_{\gamma_i} - c_{\alpha_i} b_{i_y} c_{\gamma_i}}{q_i s_{\gamma_i}} \right] \end{array} \right\} \tag{35}$$

4.3.1 Formulation of $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}\mathbf{A}$

The derivatives of $\mathbf{B}^{-1}$ are simple and can be written as:

$$\frac{\partial \mathbf{B}^{-1}}{\partial x_i} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \frac{\partial([\mathbf{p}\times])}{\partial x_i} & \mathbf{0} \end{bmatrix} \tag{36}$$

The last three derivatives of [**p**×] with respect to the orientation angles of the moving platform are three null matrices.

Let **T1** be the three dimensional matrix $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}\mathbf{A}$ and **T1k** be the *k*th plane of this matrix, k = 1, ..., 6.

The first three planes of **T1** are given by:

$$\left\{ \begin{array}{l} \mathbf{T11}^i = [0 \quad 0 \quad 0 \quad 0 \quad \cos(\gamma_i) \quad -\cos(\beta_i)] \\ \mathbf{T12}^i = [0 \quad 0 \quad 0 \quad -\cos(\gamma_i) \quad 0 \quad \cos(\alpha_i)] \\ \mathbf{T13}^i = [0 \quad 0 \quad 0 \quad \cos(\beta_i) \quad -\cos(\alpha_i) \quad 0] \end{array} \right\}^t \tag{37}$$

The last three planes of $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}\mathbf{A}$, i.e. **T14** **T15** and **T16**, are 6×6 null matrices. The superscript, i, indicates that Eq. (37) gives the expressions for the *i*th column of $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}\mathbf{A}$, i = 1, ..., 6. The special form of **T11**, **T12**, and **T13** shows that the lines of $\frac{d\mathbf{B}^{-1}}{d\mathbf{x}}\mathbf{A}$ are lines at infinity since the first three Plücker coordinates are zero [Hunt, 1978].

4.3.2 Formulating the expressions of $\mathbf{B}^{-1}\frac{d\mathbf{A}}{d\mathbf{x}_i}$

According to Eqs. (18) and (11) we obtain the following expressions for the *i*th column of $\mathbf{B}^{-1}\frac{\partial\mathbf{A}}{\partial\mathbf{x}_i}$.

Let **T2** be the three dimensional matrix $\mathbf{B}^{-1}\frac{d\mathbf{A}}{d\mathbf{x}}$. We refer to the *k*th plane of this matrix, $\mathbf{B}^{-1}\frac{\partial\mathbf{A}}{\partial\mathbf{x}_k}$, by the abbreviation **T2k** where k = 1, ..., 6. The expressions of **T21** through **T26** are given in the appendix.

By substituting the expressions of the *i*th columns of **T1k** and **T2k**, k, i = 1, ..., 6 in Eq. (33) one can see that Eq. (33) is fulfilled. This means that the lines of **T1** and the lines of **T2** intersect each other. This completes the proof that the derivatives of $\mathbf{J}^t$ with respect to position variables are groups of lines. In total we obtained 36 lines divided to six line-sextuplets with each line-sextuplet representing the derivative of $\mathbf{J}^t$ with respect to one position/orientation variable of the moving platform.

4.4 Simulation results

Numerical and graphical simulations are given below in order to visualize the results. Figure 5 shows the lines of the Jacobian matrix with arrows indicating the direction of the internal forces of the linear actuators. The dotted lines in Fig. 5 are the lines of the derivative of $\mathbf{J}^t$ with respect to the x coordinate of the moving platform.

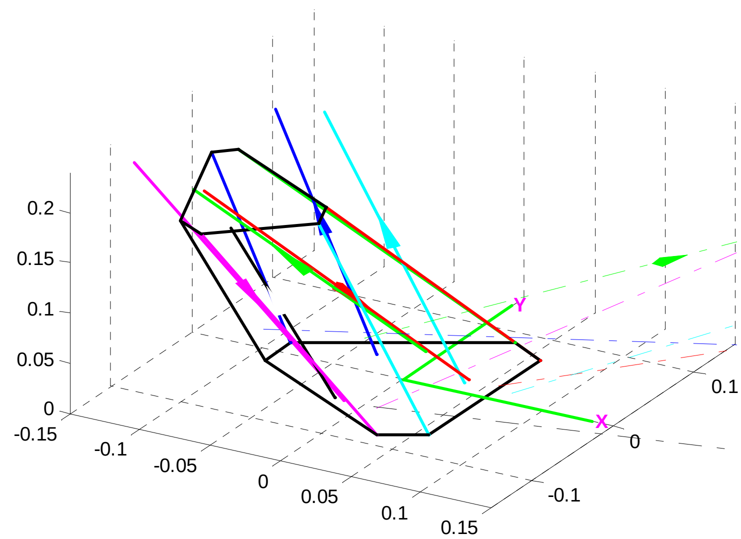

Figure 5: The lines of the Jacobian and the lines of its derivative with respect to x coordinate.

Numerical example:

The following are numerical results of a simulation of the Stewart-Gough 6–6 platform with a moving platform and a base platform having radii of 0.05 and 0.09 m, respectively. The moving platform is positioned at $\mathbf{p} = [-0.1 \quad -0.02 \quad 0.16]^t$ and rotated 30 degrees about the axis [1, 1, 1] relative to the Cartesian coordinate system in Fig. 5. The inverse Jacobian matrix is given by:

$$\mathbf{J}^t = \begin{matrix} -0.5742 & -0.6348 & -0.2662 & -0.1886 & -0.6702 & -0.5792 \\ -0.3223 & -0.2715 & -0.0610 & -0.3012 & 0.0799 & 0.3001 \\ 0.7526 & 0.7234 & 0.9620 & 0.9347 & 0.7379 & 0.7579 \\ 0.0154 & 0.0322 & 0.0245 & -0.0441 & -0.0349 & 0.0109 \\ -0.0269 & 0.0070 & 0.0317 & 0.0196 & 0.0107 & -0.0270 \\ 0.0002 & 0.0309 & 0.0088 & -0.0026 & -0.0328 & 0.0190 \end{matrix}$$

The derivatives of $\mathbf{J}^t$ with respect to e.g. x, and θ_y, are:

$$\frac{\partial(\mathbf{J}^t)}{\partial x} = \begin{matrix} 3.3431 & 2.4014 & 4.9488 & 5.8132 & 2.7368 & 3.4710 \\ -0.9232 & -0.6932 & -0.0866 & -0.3424 & 0.2661 & 0.9080 \\ 2.1555 & 1.8473 & 1.3640 & 1.0626 & 2.4570 & 2.2932 \\ 0.0440 & 0.0823 & 0.0348 & -0.0501 & -0.1161 & 0.0330 \\ -0.1226 & 0.0976 & 0.1547 & -0.0075 & -0.0213 & -0.1594 \\ -0.1208 & -0.0703 & -0.1163 & 0.2719 & 0.1316 & 0.0131 \end{matrix}$$

$$\frac{\partial(\mathbf{J}^t)}{\partial \theta_y} = \begin{matrix} -0.1226 & 0.0976 & 0.1547 & -0.0075 & -0.0213 & -0.1594 \\ -0.0433 & 0.0076 & 0.0103 & 0.0355 & -0.0043 & 0.0423 \\ -0.1121 & 0.0885 & 0.0435 & 0.0099 & -0.0189 & -0.1386 \\ -0.0169 & 0.0105 & 0.0032 & 0.0057 & 0.0005 & 0.0135 \\ 0.0373 & -0.0272 & -0.0252 & 0.0011 & 0.0059 & 0.0474 \\ 0.0041 & -0.0092 & -0.0054 & 0.0004 & -0.0019 & -0.0011 \end{matrix}$$

It is easy to see, using Eqs. (5) and (33), that the columns of $\mathbf{J}^t$ and its derivatives intersect each other and that the columns of the derivatives of $\mathbf{J}^t$ are a group of lines.

Conclusions

It is well known that the Jacobian matrix of robot manipulators is composed of Plücker coordinates of lines. In particular, in a linearly actuated fully parallel manipulator the lines are aligned with the extensible links. This paper derived analytically the expression of the derivatives of the Jacobian matrix of a six-degrees-of-freedom fully parallel manipulator. These derivatives were taken with respect to the moving platform's position/orientation variables rather than time or active joints' variables. We proved that these derivatives are also composed of lines that intersect the lines of the Jacobian matrix at the spherical joints at the base. In total, we obtained 36 lines constituted from six line-sextuplets with each line-sextuplet representing the derivative of $\mathbf{J}^t$ with respect to one position/orientation variable of the moving platform. The authors believe that interpreting geometrically the Jacobian matrix derivative as being also lines and their relation with the original Jacobian matrix lines, will facilitate the geometrical interpretation of rigidity, stability and dynamics that requires expression of the derivative of the Jacobian matrix. Further work is being done to investigate the extent of the affect of the derivative of the Jacobian on the rigidity of robots having low rigidity constants.

Appendix

The following equations give the explicit expression of the *i*th column of T2k, k,i=1, ..., 6.

$$T21=\left[\frac{s_{\alpha_i}^2}{q_i},-\frac{c_{\alpha_i}c_{\beta_i}}{q_i},-\frac{c_{\alpha_i}c_{\gamma_i}}{q_i},-\frac{p_z c_{\alpha_i}c_{\beta_i}}{q_i}+\frac{p_y c_{\alpha_i}c_{\gamma_i}}{q_i}+\frac{b_{i_z}c_{\alpha_i}c_{\beta_i}}{q_i}-\frac{b_{i_y}c_{\alpha_i}c_{\gamma_i}}{q_i},-\frac{p_z s_{\alpha_i}^2}{q_i}-\frac{p_x c_{\alpha_i}c_{\gamma_i}}{q_i}+\frac{b_{i_z}s_{\alpha_i}^2}{q_i}+\frac{b_{i_x}c_{\alpha_i}c_{\gamma_i}}{q_i},\frac{p_y s_{\alpha_i}^2}{q_i}+\frac{p_x c_{\alpha_i}c_{\beta_i}}{q_i}-\frac{b_{i_y}s_{\alpha_i}^2}{q_i}-\frac{b_{i_x}c_{\alpha_i}c_{\beta_i}}{q_i}\right]$$

$$T22=\left[-\frac{c_{\alpha_i}c_{\beta_i}}{q_i},\frac{s_{\beta_i}^2}{q_i},-\frac{c_{\beta_i}c_{\gamma_i}}{q_i},\frac{p_z s_{\beta_i}^2}{q_i}+\frac{p_y c_{\beta_i}c_{\gamma_i}}{q_i}-\frac{b_{i_z}s_{\beta_i}^2}{q_i}-\frac{b_{i_y}c_{\beta_i}c_{\gamma_i}}{q_i},\frac{p_z c_{\alpha_i}c_{\beta_i}}{q_i}-\frac{p_x c_{\beta_i}c_{\gamma_i}}{q_i}-\frac{b_{i_z}c_{\alpha_i}c_{\beta_i}}{q_i}+\frac{b_{i_x}c_{\beta_i}c_{\gamma_i}}{q_i},-\frac{p_y c_{\alpha_i}c_{\beta_i}}{q_i}-\frac{p_x s_{\beta_i}^2}{q_i}+\frac{b_{i_y}c_{\alpha_i}c_{\beta_i}}{q_i}+\frac{b_{i_x}s_{\beta_i}^2}{q_i}\right]$$

$$T23=\left[-\frac{c_{\alpha_i}c_{\gamma_i}}{q_i},-\frac{c_{\beta_i}c_{\gamma_i}}{q_i},\frac{s_{\gamma_i}^2}{q_i},-\frac{p_z c_{\beta_i}c_{\gamma_i}}{q_i}-\frac{p_y s_{\gamma_i}^2}{q_i}+\frac{b_{i_z}c_{\beta_i}c_{\gamma_i}}{q_i}+\frac{b_{i_y}s_{\gamma_i}^2}{q_i},\frac{p_z c_{\alpha_i}c_{\gamma_i}}{q_i}+\frac{p_x s_{\gamma_i}^2}{q_i}-\frac{b_{i_z}c_{\alpha_i}c_{\gamma_i}}{q_i}-\frac{b_{i_x}s_{\gamma_i}^2}{q_i},-\frac{p_y c_{\alpha_i}c_{\gamma_i}}{q_i}+\frac{p_x c_{\beta_i}c_{\gamma_i}}{q_i}+\frac{b_{i_y}c_{\alpha_i}c_{\gamma_i}}{q_i}-\frac{b_{i_x}c_{\beta_i}c_{\gamma_i}}{q_i}\right]$$

$$\begin{aligned}T24=\Bigg[&-\frac{c_{\alpha_i}p_z c_{\beta_i}-c_{\alpha_i}p_y c_{\gamma_i}+c_{\alpha_i}b_{i_y}c_{\gamma_i}-c_{\alpha_i}b_{i_z}c_{\beta_i}}{q_i},-\frac{q_i c_{\gamma_i}-p_z s_{\beta_i}^2+b_{i_z}s_{\beta_i}^2-c_{\beta_i}p_y c_{\gamma_i}+c_{\beta_i}b_{i_y}c_{\gamma_i}}{q_i},-\frac{-q_i c_{\beta_i}+p_y s_{\gamma_i}^2-b_{i_y}s_{\gamma_i}^2+c_{\gamma_i}p_z c_{\beta_i}-c_{\gamma_i}b_{i_z}c_{\beta_i}}{q_i},\\
&-\frac{p_z\left(q_i c_{\gamma_i}-p_z s_{\beta_i}^2+b_{i_z}s_{\beta_i}^2-c_{\beta_i}p_y c_{\gamma_i}+c_{\beta_i}b_{i_y}c_{\gamma_i}\right)}{q_i}+\frac{p_y\left(-q_i c_{\beta_i}+p_y s_{\gamma_i}^2-b_{i_y}s_{\gamma_i}^2+c_{\gamma_i}p_z c_{\beta_i}-c_{\gamma_i}b_{i_z}c_{\beta_i}\right)}{q_i}+\frac{b_{i_z}\left(q_i c_{\gamma_i}-p_z s_{\beta_i}^2+b_{i_z}s_{\beta_i}^2-c_{\beta_i}p_y c_{\gamma_i}+c_{\beta_i}b_{i_y}c_{\gamma_i}\right)}{q_i}\\
&-\frac{b_{i_y}\left(-q_i c_{\beta_i}+p_y s_{\gamma_i}^2-b_{i_y}s_{\gamma_i}^2+c_{\gamma_i}p_z c_{\beta_i}-c_{\gamma_i}b_{i_z}c_{\beta_i}\right)}{q_i},\frac{p_z(c_{\alpha_i}p_z c_{\beta_i}-c_{\alpha_i}p_y c_{\gamma_i}+c_{\alpha_i}b_{i_y}c_{\gamma_i}-c_{\alpha_i}b_{i_z}c_{\beta_i})}{q_i}-\frac{p_x\left(-q_i c_{\beta_i}+p_y s_{\gamma_i}^2-b_{i_y}s_{\gamma_i}^2+c_{\gamma_i}p_z c_{\beta_i}-c_{\gamma_i}b_{i_z}c_{\beta_i}\right)}{q_i}\\
&-\frac{b_{i_z}(c_{\alpha_i}p_z c_{\beta_i}-c_{\alpha_i}p_y c_{\gamma_i}+c_{\alpha_i}b_{i_y}c_{\gamma_i}-c_{\alpha_i}b_{i_z}c_{\beta_i})}{q_i}+\frac{b_{i_x}\left(-q_i c_{\beta_i}+p_y s_{\gamma_i}^2-b_{i_y}s_{\gamma_i}^2+c_{\gamma_i}p_z c_{\beta_i}-c_{\gamma_i}b_{i_z}c_{\beta_i}\right)}{q_i},-\frac{p_y(c_{\alpha_i}p_z c_{\beta_i}-c_{\alpha_i}p_y c_{\gamma_i}+c_{\alpha_i}b_{i_y}c_{\gamma_i}-c_{\alpha_i}b_{i_z}c_{\beta_i})}{q_i}\\
&+\frac{p_x\left(q_i c_{\gamma_i}-p_z s_{\beta_i}^2+b_{i_z}s_{\beta_i}^2-c_{\beta_i}p_y c_{\gamma_i}+c_{\beta_i}b_{i_y}c_{\gamma_i}\right)}{q_i}+\frac{b_{i_y}(c_{\alpha_i}p_z c_{\beta_i}-c_{\alpha_i}p_y c_{\gamma_i}+c_{\alpha_i}b_{i_y}c_{\gamma_i}-c_{\alpha_i}b_{i_z}c_{\beta_i})}{q_i}-\frac{b_{i_x}\left(q_i c_{\gamma_i}-p_z s_{\beta_i}^2+b_{i_z}s_{\beta_i}^2-c_{\beta_i}p_y c_{\gamma_i}+c_{\beta_i}b_{i_y}c_{\gamma_i}\right)}{q_i}\Bigg]\end{aligned}$$

$$\begin{aligned}T25=\Bigg[&-\frac{-q_i c_{\gamma_i}+p_z s_{\alpha_i}^2-b_{i_z}s_{\alpha_i}^2+c_{\alpha_i}p_x c_{\gamma_i}-c_{\alpha_i}b_{i_x}c_{\gamma_i}}{q_i},-\frac{-c_{\alpha_i}p_z c_{\beta_i}+c_{\beta_i}p_x c_{\gamma_i}+c_{\alpha_i}b_{i_z}c_{\beta_i}-c_{\beta_i}b_{i_x}c_{\gamma_i}}{q_i},-\frac{q_i c_{\alpha_i}-p_x s_{\gamma_i}^2+b_{i_x}s_{\gamma_i}^2-c_{\gamma_i}p_z c_{\alpha_i}+c_{\gamma_i}b_{i_z}c_{\alpha_i}}{q_i},\\
&-\frac{p_z(-c_{\alpha_i}p_z c_{\beta_i}+c_{\beta_i}p_x c_{\gamma_i}+c_{\alpha_i}b_{i_z}c_{\beta_i}-c_{\beta_i}b_{i_x}c_{\gamma_i})}{q_i}+\frac{p_y\left(q_i c_{\alpha_i}-p_x s_{\gamma_i}^2+b_{i_x}s_{\gamma_i}^2-c_{\gamma_i}p_z c_{\alpha_i}+c_{\gamma_i}b_{i_z}c_{\alpha_i}\right)}{q_i}+\frac{b_{i_z}(-c_{\alpha_i}p_z c_{\beta_i}+c_{\beta_i}p_x c_{\gamma_i}+c_{\alpha_i}b_{i_z}c_{\beta_i}-c_{\beta_i}b_{i_x}c_{\gamma_i})}{q_i}\\
&-\frac{b_{i_y}\left(q_i c_{\alpha_i}-p_x s_{\gamma_i}^2+b_{i_x}s_{\gamma_i}^2-c_{\gamma_i}p_z c_{\alpha_i}+c_{\gamma_i}b_{i_z}c_{\alpha_i}\right)}{q_i},\frac{p_z\left(-q_i c_{\gamma_i}+p_z s_{\alpha_i}^2-b_{i_z}s_{\alpha_i}^2+c_{\alpha_i}p_x c_{\gamma_i}-c_{\alpha_i}b_{i_x}c_{\gamma_i}\right)}{q_i}-\frac{p_x\left(q_i c_{\alpha_i}-p_x s_{\gamma_i}^2+b_{i_x}s_{\gamma_i}^2-c_{\gamma_i}p_z c_{\alpha_i}+c_{\gamma_i}b_{i_z}c_{\alpha_i}\right)}{q_i}\\
&-\frac{b_{i_z}\left(-q_i c_{\gamma_i}+p_z s_{\alpha_i}^2-b_{i_z}s_{\alpha_i}^2+c_{\alpha_i}p_x c_{\gamma_i}-c_{\alpha_i}b_{i_x}c_{\gamma_i}\right)}{q_i}+\frac{b_{i_x}\left(q_i c_{\alpha_i}-p_x s_{\gamma_i}^2+b_{i_x}s_{\gamma_i}^2-c_{\gamma_i}p_z c_{\alpha_i}+c_{\gamma_i}b_{i_z}c_{\alpha_i}\right)}{q_i},-\frac{p_y\left(-q_i c_{\gamma_i}+p_z s_{\alpha_i}^2-b_{i_z}s_{\alpha_i}^2+c_{\alpha_i}p_x c_{\gamma_i}-c_{\alpha_i}b_{i_x}c_{\gamma_i}\right)}{q_i}\\
&+\frac{p_x(-c_{\alpha_i}p_z c_{\beta_i}+c_{\beta_i}p_x c_{\gamma_i}+c_{\alpha_i}b_{i_z}c_{\beta_i}-c_{\beta_i}b_{i_x}c_{\gamma_i})}{q_i}+\frac{b_{i_y}\left(-q_i c_{\gamma_i}+p_z s_{\alpha_i}^2-b_{i_z}s_{\alpha_i}^2+c_{\alpha_i}p_x c_{\gamma_i}-c_{\alpha_i}b_{i_x}c_{\gamma_i}\right)}{q_i}-\frac{b_{i_x}(-c_{\alpha_i}p_z c_{\beta_i}+c_{\beta_i}p_x c_{\gamma_i}+c_{\alpha_i}b_{i_z}c_{\beta_i}-c_{\beta_i}b_{i_x}c_{\gamma_i})}{q_i}\Bigg]\end{aligned}$$

$$
\begin{aligned}
T26 = \Bigg[& -\frac{q_i c_{\beta_i} - p_y s_{\alpha_i}^2 + b_{i_y} s_{\alpha_i}^2 - c_{\alpha_i} p_x c_{\beta_i} + c_{\alpha_i} b_{i_x} c_{\beta_i}}{q_i}, -\frac{-q_i c_{\alpha_i} + p_x s_{\beta_i}^2 - b_{i_x} s_{\beta_i}^2 + c_{\beta_i} p_y c_{\alpha_i} - c_{\beta_i} b_{i_y} c_{\alpha_i}}{q_i}, -\frac{c_{\alpha_i} p_y c_{\gamma_i} - c_{\beta_i} p_x c_{\gamma_i} + c_{\beta_i} b_{i_x} c_{\gamma_i} - c_{\alpha_i} b_{i_y} c_{\gamma_i}}{q_i}, \\
& -\frac{p_z \left(-q_i c_{\alpha_i} + p_x s_{\beta_i}^2 - b_{i_x} s_{\beta_i}^2 + c_{\beta_i} p_y c_{\alpha_i} - c_{\beta_i} b_{i_y} c_{\alpha_i}\right)}{q_i} + \frac{p_y (c_{\alpha_i} p_y c_{\gamma_i} - c_{\beta_i} p_x c_{\gamma_i} + c_{\beta_i} b_{i_x} c_{\gamma_i} - c_{\alpha_i} b_{i_y} c_{\gamma_i})}{q_i} + \frac{b_{i_z} \left(-q_i c_{\alpha_i} + p_x s_{\beta_i}^2 - b_{i_x} s_{\beta_i}^2 + c_{\beta_i} p_y c_{\alpha_i} - c_{\beta_i} b_{i_y} c_{\alpha_i}\right)}{q_i} \\
& -\frac{b_{i_y} (c_{\alpha_i} p_y c_{\gamma_i} - c_{\beta_i} p_x c_{\gamma_i} + c_{\beta_i} b_{i_x} c_{\gamma_i} - c_{\alpha_i} b_{i_y} c_{\gamma_i})}{q_i}, \frac{p_z \left(q_i c_{\beta_i} - p_y s_{\alpha_i}^2 + b_{i_y} s_{\alpha_i}^2 - c_{\alpha_i} p_x c_{\beta_i} + c_{\alpha_i} b_{i_x} c_{\beta_i}\right)}{q_i} - \frac{p_x (c_{\alpha_i} p_y c_{\gamma_i} - c_{\beta_i} p_x c_{\gamma_i} + c_{\beta_i} b_{i_x} c_{\gamma_i} - c_{\alpha_i} b_{i_y} c_{\gamma_i})}{q_i} \\
& -\frac{b_{i_z} \left(q_i c_{\beta_i} - p_y s_{\alpha_i}^2 + b_{i_y} s_{\alpha_i}^2 - c_{\alpha_i} p_x c_{\beta_i} + c_{\alpha_i} b_{i_x} c_{\beta_i}\right)}{q_i} + \frac{b_{i_x} (c_{\alpha_i} p_y c_{\gamma_i} - c_{\beta_i} p_x c_{\gamma_i} + c_{\beta_i} b_{i_x} c_{\gamma_i} - c_{\alpha_i} b_{i_y} c_{\gamma_i})}{q_i}, -\frac{p_y \left(q_i c_{\beta_i} - p_y s_{\alpha_i}^2 + b_{i_y} s_{\alpha_i}^2 - c_{\alpha_i} p_x c_{\beta_i} + c_{\alpha_i} b_{i_x} c_{\beta_i}\right)}{q_i} \\
& +\frac{p_x \left(-q_i c_{\alpha_i} + p_x s_{\beta_i}^2 - b_{i_x} s_{\beta_i}^2 + c_{\beta_i} p_y c_{\alpha_i} - c_{\beta_i} b_{i_y} c_{\alpha_i}\right)}{q_i} + \frac{b_{i_y} \left(q_i c_{\beta_i} - p_y s_{\alpha_i}^2 + b_{i_y} s_{\alpha_i}^2 - c_{\alpha_i} p_x c_{\beta_i} + c_{\alpha_i} b_{i_x} c_{\beta_i}\right)}{q_i} \\
& -\frac{b_{i_x} \left(-q_i c_{\alpha_i} + p_x s_{\beta_i}^2 - b_{i_x} s_{\beta_i}^2 + c_{\beta_i} p_y c_{\alpha_i} - c_{\beta_i} b_{i_y} c_{\alpha_i}\right)}{q_i} \Bigg]
\end{aligned}
$$

References

Ben-Horin, R., *Criteria for Analysis of Parallel Robots*, Ph.D. dissertation, The Technion, Israel, 1997.

Byung-Ju, Y., Freeman, R., Tesar, D., “Open-Loop Stiffness Control of Overconstrained Mechanisms/Robotic Linkage Systems.” IEEE International Conference on *Robotics and Automation*, pp. 1340-1345, 1989.

Cleary, C., Uebel, M., “Jacobian Formulation For A Novel 6-DOF Parallel Manipulator.” IEEE International Conference on *Robotics and Automation*, 1994, Vol.3, pp.2377-2382.

Collins, C. L., Long. G. L., “The Singularity Analysis of an In-Parallel Hand Controller for Force-Reflected Teleoperation.” IEEE Transactions on *Robotics and Automation*, Vol. 11, No. 5, 1995, pp. 661-669.

Dandurand, A., “The Rigidity of Compound Spatial Grid,” *Structural Topology*, Vol. 10, pp. 41-56, 1984.

Dutré, S., Bruyninckx, H., De Schutter, J., “The analytical Jacobian and its derivative for a parallel manipulator,” IEEE International Conference on *Robotics and Automation*, pp. 2961-2966, 1997.

Graustein, W. C., *Introduction to Higher Geometry*, The Macmillan Company, 1930.

Hunt, K. H., *Kinematic Geometry of Mechanisms*. Clarendon Press, Oxford, 1978.

Hunt, K. H., Samuel, A. E., McAree, P. R., “Special Configurations of Multi Freedom Grippers – A kinematic Study,” The International Journal of *Robotics Research*, Vol. 10, No. 2, pp. 123-134, 1991.

Kock, S., Schumacher, W., "A parallel x-y Manipulator with Actuation Redundancy for High-Speed and Active-Stiffness Applications.", IEEE International Conference on *Robotics and Automation*, Vol. 2, pp. 2295-2300, 1998.

Ma, O., Angeles, J., "Architecture Singularities of Parallel Manipulators," International Journal of *Robotics and Automation*, Vol. 7, No. 1, 1992.

Merlet, J. P., "Singular Configurations of Parallel Manipulators and Grassmann Geometry," The International Journal of *Robotics Research*, Vol. 8, No. 5, 1989.

Merlet, J. P. and Gosselin, C., "NOUVELLE ARCHITECTURE POUR UN MANIPULATEUR PARALLELE A SIX DEGRES DE LIBERTE." *Mechanism and Machine Theory*, Vol. 26, No. 1, pp. 77-90, 1991.

Pellegrini, M., "Ray Shooting and Lines in Space," *Handbook of Discrete and Computational Geometry*, Goodman, J., O'Rourke, J., Eds., CRC Press, pp. 599-612, 1997.

Pottman, H., Peternell, M., Ravani, B,. "An introduction to line geometry with applications," Computer-Aided Design, Vol. 31, pp. 3-16, 1999.

Simaan, N., *Analysis and Synthesis of Parallel Robots for Medical Applications*, Master Thesis, Technion, Israel, 1999.

Simaan, N., Glozman, D., Shoham, M., "Design Considerations of New Six Degrees-Of-Freedom Parallel Robots." IEEE International Conference on *Robotics and Automation*, Vol. 2, pp. 1327-1333, 1998.

Sommerville, D. M. Y., *Analytical Geometry of Three Dimensions*, Cambridge Press, 1934.

Tsai, L-W., "The Jacobian Analysis of Parallel Manipulators Using Reciprocal Screws," Advances in Robot Kinematics: Analysis and Control, Lenarčič, J., and Husty, M. L., eds., pp. 327-336, Kluwer Academic Publishers, 1998.

Yoshikawa, T., *Foundation of Robotics Analysis and Control*, MIT Press, 1990.