

DUAL NUMBERS REPRESENTATION OF RIGID BODY DYNAMICS

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Abstract: A three-dimensional representation of rigid body dynamic equations becomes possible by introducing the dual inertia operator. This paper generalizes this result and by using motor transformation rules and the dual inertia operator, gives a general expression for the three-dimensional dynamic equation of a rigid body with respect to an arbitrary point.

Then, the dual Lagrange equation is formulated by developing derivative rules of a real function with respect to dual variables. It is shown that the same rules hold for derivatives of a real function with respect to both real and dual variables.

The analogy between rigid body spherical dynamics and the dual three-dimensional spatial one is discussed and summarized.

1. INTRODUCTION

Dual numbers were introduced in the 19th century by Clifford [1], and their application to rigid body kinematics was subsequently generalized by Kotelnikov and Study in their principle of transference [2-6]. The principle of transference states that if dual numbers replace real ones, then all relations of vector algebra for intersecting lines are also valid for skew lines. In practice, this means that all rules of vector algebra for the kinematics of a rigid body with a fixed point (spherical kinematics) also hold for motor algebra of a free rigid body (spatial kinematics). As a result, *a general rigid body motion can be described by only three dual equations rather than six real ones.*

For several decades there were attempts to apply dual numbers to rigid body dynamics. Investigators showed that the momentum of a rigid body can be described as a motor that obeys the motor transformation rule; hence, its derivative with respect to time yields the dual force. However, in those investigations, while going from the velocity motor to the momentum motor, there was always a need to expand the equation to six dimensions and to treat the velocity motor as two separate real vectors. This process actually diminishes one of the main advantages of dual numbers - namely, compactness of representation.

The recently introduced dual inertia operator [7] suggests a solution to this problem. By giving the mass a dual property, results have shown that it is possible to remain in a three-dimensional dual space throughout the entire derivation of the dynamic equations. Also, unlike the result obtained with the binor inertia matrix, which requires an artificial permutation of the dual and the real parts of the

momentum, the dual inertia operator yields the equations in a correct order. Thus, the dual inertia operator can be perceived as a necessary link between the velocity and the momentum motors.

This paper expands, in its first part, the results reported in Brodsky and Shoham's paper [7] and gives a more general expression for the dynamic equations which enables the calculation of the dual momentum and the dual force with respect to an arbitrary point. Then, the moment-product is applied to obtain the energy of a rigid body. Its mathematical properties, i.e., analyticity and derivative rules of a real function with respect to dual variables are developed and subsequently applied to derive Lagrange's formulation of equations-of-motion in a complete three-dimensional dual form. Finally, the analogy between the spherical and the spatial dynamics is discussed and summarized in a table.

1.1 Dual Numbers and Motors

We start our discussion by reviewing some of the basic concepts of dual numbers.

Dual numbers were introduced by Clifford [1] as a necessary unit for motor algebra (see

$$\hat{a} = a + \varepsilon a^\circ, \quad (1)$$

where ε is the dual unit with multiplication rule

$$\varepsilon^2 = 0. \quad (2)$$

The algebra of dual numbers results from this definition. Two dual numbers are equal if and only if their real and dual parts are equal, respectively. As in the case of complex numbers, addition of two dual numbers requires separate addition of their real and dual parts:

$$(x + \varepsilon x^\circ) + (y + \varepsilon y^\circ) = (x + y) + \varepsilon(x^\circ + y^\circ). \quad (3)$$

Multiplication of two dual numbers results in:

$$(x + \varepsilon x^\circ)(y + \varepsilon y^\circ) = xy + \varepsilon(x^\circ y + xy^\circ). \quad (4)$$

Division of dual numbers \hat{a}/\hat{b} is defined as the inverse operation to multiplication. Due to the special property of dual numbers, Eq. (2), division is possible and unambiguous only if $b \neq 0$

$$\frac{\hat{a}}{\hat{b}} = \frac{a}{b} + \varepsilon \left(\frac{a^\circ}{b} - \frac{ab^\circ}{b^2} \right), \quad b \neq 0. \quad (5)$$

Clifford also coined the term 'motor' which can be defined as a combination of three-dimensional rotor (line-vector), \mathbf{a} , and vector (free-vector), \mathbf{a}° , that refers to some point [1, 2],

$$\hat{\mathbf{a}} = \mathbf{a} + \varepsilon \mathbf{a}^\circ, \quad (6)$$

and satisfies the motor transformation rule, [8], the matrix form of which can be written as follows:

$$\mathbf{a} + \varepsilon \mathbf{a}_O^\circ = \hat{\mathbf{D}}_{OA} (\mathbf{a} + \varepsilon \mathbf{a}_A^\circ), \quad (7)$$

where $\hat{\mathbf{D}}_{OA}$ is an Hermitian matrix

$$\hat{\mathbf{D}}_{OA} = [\mathbf{1} + \varepsilon(\mathbf{d}_{OA} \times)] = \begin{bmatrix} 1 & -\varepsilon d_{OA,z} & \varepsilon d_{OA,y} \\ \varepsilon d_{OA,z} & 1 & -\varepsilon d_{OA,x} \\ -\varepsilon d_{OA,y} & \varepsilon d_{OA,x} & 1 \end{bmatrix}. \quad (8)$$

This matrix can be interpreted in two ways: first, as an operator that derives motor components at point O from a known motor at point A; and second, as an operator that refers a motor given in a system with origin at point A, to a parallel system with origin at point O.

Transformation of a motor back from point O to A, i.e., the inverse operation to $\hat{\mathbf{D}}_{OA}$, is given by the transpose (also the conjugate) of the above matrix:

$$\hat{\mathbf{D}}_{AO} = \hat{\mathbf{D}}_{OA}^{-1} = \hat{\mathbf{D}}_{OA}^T = \hat{\mathbf{D}}_{OA}^* \quad (9)$$

The motor transformation rule is used in the following sections to refer dual vectors of velocity, force and momentum to an arbitrary reference point.

Every motor can be brought to such a reference point that its rotor and vector parts become parallel, which turns a motor into an equivalent screw. A parallel line through this point is the screw axis.

Geometrically, when motors are perceived as screws, then there exists a dual angle

$$\hat{\theta} = \theta + \varepsilon d, \quad (10)$$

which is a combination of an angle, θ , and a distance, d , between the skew axes of their respective screws [3].

With these definitions, scalar and vector multiplication of dual vectors have exactly the same pattern as those of real ones. The dual form, however, includes information not only about the relative orientation between the free vectors but also about their relative position. This property stems from the principle of transference formulated by Kotelnikov in 1895 and Study [3] and investigated further by Rooney [4], Martinez and Duffy [6] and Chevallier [9].

Dimentberg [2] formulated it essentially in the following way: "...at least all the formulas of vector algebra... will serve as formulas for motor algebra...; here the complex modulus of the screw will correspond to the modulus of the vector in the new formulas, and the complex (dual) angle between the axes of two motors will correspond to the angle between two vectors". The principle of transference has, in fact, been used to expand relations in spherical into spatial kinematics by replacing real with dual numbers.

Dimentberg argued that the only case when the principle of transference loses its sense is when vector moduli vanish. In this case the corresponding motors are degenerate, and such a case needs a special treatment.

1.2 Dual Function of Dual Numbers

Dual function of dual number presents a mapping of a dual numbers space on itself, namely,

$$\hat{f}(\hat{x}) = f(x, x^\circ) + \varepsilon f^\circ(x, x^\circ), \quad (11)$$

where $\hat{x} = x + x^\circ$ is a dual variable, f and f° are two, generally different, functions of two variables. This type of function is referred to simply as the dual functions throughout the work.

Properties of dual functions were thoroughly investigated by Dimentberg [2]. He derived the general expression for dual analytic (differentiable) function as follows:

$$\hat{f}(x + \varepsilon x^\circ) = f + \varepsilon f^\circ = f(x) + \varepsilon (x^\circ f'(x) + \tilde{f}(x)), \quad (12)$$

where $\tilde{f}(x)$ is an arbitrary function of a real part of a dual variable.

The analytic condition for dual function is:

$$\frac{\partial f^\circ}{\partial x^\circ} = \frac{\partial f}{\partial x}. \quad (13)$$

The derivative of such a dual function with respect to a dual variable is:

$$\frac{d\hat{f}(\hat{x})}{d\hat{x}} = \frac{\partial f^\circ}{\partial x^\circ} + \varepsilon \frac{\partial f^\circ}{\partial x}, \quad (14)$$

or, taking into account Eq. (13):

$$\frac{d\hat{f}(\hat{x})}{d\hat{x}} = \frac{\partial f}{\partial x} + \varepsilon \frac{\partial f^\circ}{\partial x} = f'(x) + \varepsilon (x^\circ f''(x) + \tilde{f}'(x)). \quad (15)$$

This definition allows us to write the dual form of functions, for example:

$$\begin{aligned} \sin(x + \varepsilon x^\circ) &= \sin(x) + \varepsilon x^\circ \cos(x), \\ \cos(x + \varepsilon x^\circ) &= \cos(x) - \varepsilon x^\circ \sin(x), \\ e^{a + \varepsilon b} &= e^a \cdot e^{\varepsilon b} = e^a (1 + \varepsilon b) = e^a + \varepsilon b e^a; \\ \sqrt{\hat{x}} &= \sqrt{x + \varepsilon x^\circ} = \sqrt{x} + \varepsilon x^\circ / 2\sqrt{x}, \quad (x > 0). \end{aligned} \quad (16)$$

In particular, trigonometric relations are maintained in dual angles.

$$\sin^2 \hat{x} + \cos^2 \hat{x} = 1, \quad e^{\hat{x} + \hat{y}} = e^{\hat{x}} \cdot e^{\hat{y}} \quad (17)$$

The above formulas have been widely applied in kinematics (see, for example, [10-19]).

One of the main goals of this paper is to develop this algebra further and to apply it also to dynamics, as is shown in the following sections.

1.3 The Inertia Binor and its Application to Rigid Body Dynamics

In the last thirty years a growing number of investigations concerning the application of dual numbers to rigid body dynamics have appeared in the literature [2, 13, 20-31].

The use of dual numbers for rigid body dynamics is justified by the fact that by combining body's linear and angular momentum vectors, \mathbf{H}_L and \mathbf{H}_A , respectively, into a dual vector

$$\hat{\mathbf{H}} = \mathbf{H}_L + \varepsilon \mathbf{H}_A, \quad (18)$$

one obtains a motor, which satisfies the motor transformation rule, Eq. (7).

The common approach to dual number formulation of rigid body dynamics is based on Kislitcin's inertia binor [2]. The binor is a six-by-six matrix of the body's mass and its first and second moment of mass:

$$(\mathbf{T}) = \begin{bmatrix} 0 & S_z & -S_y & m & 0 & 0 \\ -S_z & 0 & S_x & 0 & m & 0 \\ S_y & -S_x & 0 & 0 & 0 & m \\ I_{xx} & I_{xy} & I_{xz} & 0 & -S_z & S_y \\ I_{xy} & I_{yy} & I_{yz} & S_z & 0 & -S_x \\ I_{xz} & I_{yz} & I_{zz} & -S_y & S_x & 0 \end{bmatrix}. \quad (19)$$

The inertia binor is analogous to the 6x6 inertia matrix introduced by Von Mises [31].

Multiplying the binor by the velocity motor, one obtains a six-dimensional motor of momentum

$$\begin{bmatrix} \mathbf{H}_L \\ \mathbf{H}_A \end{bmatrix} = (\mathbf{T}) \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_c \end{bmatrix}. \quad (20)$$

the first three components of which form the linear and the other three form the angular momentum. This six-dimensional vector is, in turn, combined into three-dimensional dual vector of rigid body momentum, Eq. (18). It is clear that one of the binor's drawbacks is that it operates on a six-dimensional vector rather than on a three-dimensional velocity motor. A similar approach, consisting of a separate calculation of real (linear) and dual (angular) parts of rigid body dual momentum, was used by Yang [20, 21].

Using binor inertia matrix for evaluating the dynamic equations, Dimentberg has come to the conclusion that one could not obtain this matrix from any real operator by substituting dual quantities for real counterparts, i.e. by applying the principle of transference. That led Dimentberg to the conclusion "that it is impossible to obtain a screw equation of the dynamics of an arbitrary moving body from the dynamics vector equation of a body with a fixed point by application of the transfer principle."

Only recently, a new dual number formulation of a rigid body dynamics was suggested that casts the equations throughout the derivation into a three-dimensional form [7, 32]. This approach is detailed in the next section.

2. DUAL CHARACTER OF MASS

Recent papers introduce a new approach to the formulation of rigid body dynamic equations in a dual number form by giving the mass a dual character.

Rigid body is an aggregation of elementary masses, the velocity of each can be derived from the body's velocity motor $\hat{\mathbf{v}} = \boldsymbol{\omega} + \boldsymbol{\varepsilon}\mathbf{v}$ (see Fig. 1). Linear momentum of each elementary mass can be perceived as an operation of the mass on the velocity. This operation does not only multiply the velocity by a scalar, but it actually changes a velocity motor's basic property. The velocity motor, as shown by Chevallier [17], produces a vector field of velocities. When an elementary mass dm_A , situated at a specific point, having a linear velocity \mathbf{v}_A , the resulting entity, $\mathbf{v}_A dm_A$, is tied to the elementary mass at a specific location. Hence, it is no longer a free vector, as is the linear velocity, but in fact, a rotor.

Thus, the result of a body's mass operation on a vector field is an infinite set of rotors of linear momentum related to an infinite set of elementary masses (points) composing the body.

At each point of the body, the obtained rotor is proportional to the vector component of the velocity motor (linear velocity) at this point and to the elementary mass carried by this point. It is, therefore, a rotor of linear momentum of the elementary mass situated at a given point of the body.

Mathematically, this process is described by a dual mass operator, $dm \frac{d}{d\varepsilon}$, which acts on a velocity motor and changes it into a rotor

$$d\hat{m}_A \hat{v} = dm_A \frac{d}{d\varepsilon} (\boldsymbol{\omega} + \varepsilon \mathbf{v}) = \mathbf{v}_A dm_A. \quad (21)$$

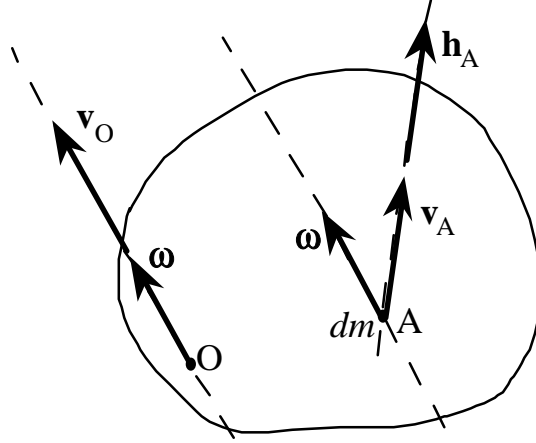


Fig. 1 Dual Momentum of a Rigid Body

The above consideration is applicable also to the force acting on a mass in a gravitational field. The gravitation forms a *vector* field. If a mass is located in this field, then a force *rotor*, which acts on the mass and is tied to its location, is developed. The dual mass operator, as defined above, operates on the gravitational vector by transforming it into a rotor proportional to the gravitational vector magnitude and aligned in its direction.

The above defined operator $\frac{d}{d\varepsilon}$ has a complementary sense of Clifford's dual unit which "when applied to any motor, changes it into a vector parallel to its axis and proportional to the rotor part of it" [1]. It does not have, however, an infinitesimal sense.

Comparing the operations of ε and $\frac{d}{d\varepsilon}$ on a dual vector, one obtains:

$$\varepsilon \hat{\mathbf{a}} = \varepsilon (\mathbf{a} + \varepsilon \mathbf{a}_0) = \varepsilon \mathbf{a}, \quad \frac{d}{d\varepsilon} \hat{\mathbf{a}} = \frac{d}{d\varepsilon} (\mathbf{a} + \varepsilon \mathbf{a}_0) = \mathbf{a}_0. \quad (22)$$

Like the dual unit ε , when the operator $\frac{d}{d\varepsilon}$ is applied twice to a dual motor it reduces this motor to zero.

With the above definition of dual mass, Newton's second law can now be written in a correct dual form:

$$\hat{\mathbf{f}} = \hat{m} \frac{d\hat{\mathbf{u}}}{dt} = m \frac{d}{d\varepsilon} \left(\varepsilon \frac{d\mathbf{v}}{dt} \right) = m \frac{d\mathbf{v}}{dt}, \quad (23)$$

where $\hat{\mathbf{f}} = \mathbf{f}$ is a force - a pure real number - acting on a particle of mass m , and $\hat{\mathbf{u}} = \varepsilon \mathbf{v}$ is its dual (linear) velocity.

3. DUAL NUMBER REPRESENTATION OF NEWTON-EULER'S DYNAMIC EQUATIONS

Consistent application of the dual mass concept allows one to calculate the dual momentum motor of a system of particles and of a rigid body about an arbitrary point.

A momentum motor of a system of particles is a sum of linear momentum rotors of each particle. To calculate this quantity, all rotors are brought to a common reference point O, using the motor transformation rule, Eq. (7), and then, rotor and vector parts are summed up, respectively:

$$\hat{\mathbf{h}}_O = \sum_{i=1}^n \hat{\mathbf{D}}_{OA_i} \hat{m}_{A_i} \hat{\mathbf{v}}_{A_i} = \sum_{i=1}^n m_{A_i} \mathbf{v}_{A_i} + \varepsilon \sum_{i=1}^n m_{A_i} \mathbf{d}_{OA_i} \times \mathbf{v}_{A_i}, \quad (24)$$

where n is the number of particles, m_{A_i} is i-th particle mass and A_i is i-th particle location.

A momentum motor of a rigid body is obtained in a similar way by integration over the body's mass:

$$\hat{\mathbf{h}}_B = \int_{A \in b} \hat{\mathbf{D}}_{BA} d\hat{m}_A \hat{\mathbf{v}}_A, \quad (25)$$

where A and B are, in general, two different points.

Note that the velocity motor $\hat{\mathbf{v}}_A$ of any point A of the body is obtained from only one velocity motor at some point, O, by using the motor transformation rule. The following expression of a momentum motor is then obtained:

$$\hat{\mathbf{h}}_B = \int_{A \in b} \hat{\mathbf{D}}_{BA} \left(dm_A \frac{d}{d\varepsilon} \left[\hat{\mathbf{D}}_{AO} (\boldsymbol{\omega} + \varepsilon \mathbf{v}_O) \right] \right) = m \mathbf{v}_O + m \mathbf{D}_{CO} \boldsymbol{\omega} + \varepsilon [\mathbf{I}_{BO} \boldsymbol{\omega} - m \mathbf{D}_{CB} \mathbf{v}_O], \quad (26)$$

where $\mathbf{D} = [\mathbf{d} \times]$ and $\mathbf{I}_{BO} = \mathbf{I}_C - m [\mathbf{D}_{CB}] [\mathbf{D}_{CO}]$, (27)

or explicitly:

$$\mathbf{I}_{BO} = \begin{bmatrix} I_{Cxx} + d_{CB_y} d_{CO_y} + d_{CB_z} d_{CO_z} & I_{Cxy} - d_{CB_y} d_{CO_x} & I_{Cxz} - d_{CB_z} d_{CO_x} \\ I_{Cxy} - d_{CB_x} d_{CO_y} & I_{Cxx} + d_{CB_y} d_{CO_y} + d_{CB_z} d_{CO_z} & I_{Cyz} - d_{CB_z} d_{CO_y} \\ I_{Cxz} - d_{CB_x} d_{CO_z} & I_{Cyz} - d_{CB_y} d_{CO_z} & I_{Cxx} + d_{CB_y} d_{CO_y} + d_{CB_z} d_{CO_z} \end{bmatrix}. \quad (28)$$

Eq. (26) describes the most general case, where the velocity motor of a rigid body is given at some point O, while the momentum is calculated at some other point B. It gives the expression of the dual momentum of a rigid body calculated with respect to an arbitrary point, in a complete three-dimensional dual form. *It utilizes the dual mass operator and motor transformation rule (does not use the inertia binor matrix) and stays three-dimensional.*

Defining the dual inertia operator as:

$$\hat{\mathbf{M}} = \hat{m} + \hat{\mathbf{I}}_C = \begin{bmatrix} m \frac{d}{d\varepsilon} + \varepsilon I_{xx} & \varepsilon I_{xy} & \varepsilon I_{xz} \\ \varepsilon I_{xy} & m \frac{d}{d\varepsilon} + \varepsilon I_{yy} & \varepsilon I_{yz} \\ \varepsilon I_{xz} & \varepsilon I_{yz} & m \frac{d}{d\varepsilon} + \varepsilon I_{zz} \end{bmatrix}, \quad (29)$$

Eq. (26) can be rewritten in a more compact form:

$$\hat{\mathbf{h}}_B = \hat{\mathbf{D}}_{BC} \hat{\mathbf{M}} (\hat{\mathbf{D}}_{CO} \hat{\mathbf{v}}_O), \quad (30)$$

where $\hat{\mathbf{D}}_{CO}$ and $\hat{\mathbf{D}}_{BC}$ are motor transformation matrices from point O to the center of mass, and from the center of mass to an arbitrary point B, respectively, as defined by Eq. (8).

If orientational changes are also considered, i.e., the inertia parameters are given with respect to the body-attached coordinate system C, while velocity, radius-vectors and momentum are measured with respect to some other coordinate system N, one obtains the dual momentum with respect to point B in coordinated system N by the following equation:

$$\hat{\mathbf{h}}_B = \hat{\mathbf{D}}_{BC} \mathbf{A}_C^N \hat{\mathbf{M}}(\mathbf{A}_N^C \hat{\mathbf{D}}_{CO} \hat{\mathbf{v}}_O) = \hat{\mathbf{A}}_C^{N,B} \hat{\mathbf{M}}(\hat{\mathbf{A}}_{N,O}^C \hat{\mathbf{v}}_O), \quad (31)$$

where \mathbf{A}_C^N is a rotation matrix from C to N, $\hat{\mathbf{A}}_C^{N,B}$ is a dual transformation matrix of rotation from C to N and translation from origin of C to point B, and $\hat{\mathbf{A}}_{N,O}^C$ is a dual transformation matrix of rotation from N to C and translation from point O to origin of C. This equation can also be obtained by dualizing the equation for angular momentum, $\mathbf{H}_O = \mathbf{A}^T \mathbf{I} \mathbf{A} \boldsymbol{\omega}$, where dual transformation matrices replace real ones, dual velocity replaces angular velocity and the dual inertia operator replaces the inertia matrix. This form is the most general expression of dual momentum about an arbitrary point, which is an extension of the expression given in Dimentberg [2], Yang [21] and Brodsky and Shoham [7].

It is more practical to obtain an expression for the dual momentum when velocity and momentum motor are referred to the same point, say A:

$$\hat{\mathbf{h}}_A = \hat{\mathbf{D}}_{AC} \hat{\mathbf{M}}(\hat{\mathbf{D}}_{CA} \hat{\mathbf{v}}_A) = m \mathbf{v}_A + m \mathbf{D}_{CA} \boldsymbol{\omega} + \varepsilon [\mathbf{I}_A \boldsymbol{\omega} - m \mathbf{D}_{CA} \mathbf{v}_A], \quad (32)$$

where $\mathbf{I}_A = \mathbf{I}_C - m[\mathbf{D}_{CA}]^2$.

A significantly simpler result is obtained in the particular case in which the momentum is calculated with respect to the center of mass C, [7].

$$\hat{\mathbf{h}}_C = \hat{\mathbf{M}} \hat{\mathbf{v}}_C. \quad (33)$$

It is interesting to note that similar to kinematics, this dynamic equation is a dual form of a real expression for angular motion.

In order to obtain dynamic equations in Newton-Euler's form, one of the above equations is used along with the time derivative rule of a motor, [2, 21, 33]:

$$\frac{d}{dt} \hat{\mathbf{h}}_A = \frac{\partial}{\partial t} \hat{\mathbf{h}}_A + \hat{\mathbf{v}}_A \times \hat{\mathbf{h}}_A. \quad (34)$$

Note that Eq. (34) also implies that the derivative of any motor expressed in this case with respect to a moving coordinate system, is also a motor.

It is interesting to give here a six-dimensional expansion (binor version) of the general case, Eq.(30):

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ L_{B,x} \\ L_{B,y} \\ L_{B,z} \end{bmatrix} = \begin{bmatrix} m & 0 & 0 & 0 & -m d_{CA,z} & m d_{CA,y} \\ 0 & m & 0 & m d_{CA,z} & 0 & -m d_{CA,x} \\ 0 & 0 & m & -m d_{CA,y} & m d_{CA,x} & 0 \\ 0 & m d_{CB,z} & -m d_{CB,y} & I_{BA,x} & I_{BA,xy} & I_{BA,nz} \\ -m d_{CB,z} & 0 & m d_{CB,x} & I_{BA,xy} & I_{BA,y} & I_{BA,yz} \\ m d_{CB,y} & -m d_{CB,x} & 0 & I_{BA,xz} & I_{BA,yz} & I_{BA,z} \end{bmatrix} \begin{bmatrix} v_{A,x} \\ v_{A,y} \\ v_{A,z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \quad (35)$$

When compared to the regular binor inertia matrix [2], the above expression has different terms at the lower 3×6 sub-matrix. The above equation, unlike the regular binor inertia, allows calculating the dual momentum with respect to a point different from that one at which the velocity motor is given.

Rigid body dynamic equations are obtained by differentiating Eq. (32) with respect to time according to Eq. (34). This results in a dual force motor (force and moment) acting at point A where momentum is expressed:

$$\hat{\mathbf{f}}_A = \frac{d}{dt} \hat{\mathbf{h}}_A = \frac{\partial}{\partial t} \hat{\mathbf{h}}_A + \hat{\mathbf{v}}_A \times \hat{\mathbf{h}}_A = \frac{\partial}{\partial t} (m\mathbf{v}_A + m\mathbf{D}_{CA} \boldsymbol{\omega}) + m\boldsymbol{\omega} \times \hat{\mathbf{v}}_A + m\boldsymbol{\omega} \times (\mathbf{D}_{CA} \boldsymbol{\omega}) + \varepsilon \left[\frac{\partial}{\partial t} (\mathbf{I}_A \boldsymbol{\omega} - m\mathbf{D}_{CA} \mathbf{v}_A) + m\mathbf{D}_{AC} (\boldsymbol{\omega} \times \mathbf{v}_A) + \boldsymbol{\omega} \times (\mathbf{I}_A \boldsymbol{\omega}) \right]. \quad (36)$$

To obtain the dual force acting on the body at the center of mass, we differentiate Eq. (33) with respect to time in center of the mass body-fixed coordinate system, which leads to a much simpler expression:

$$\hat{\mathbf{f}}_C = \frac{d}{dt} \hat{\mathbf{h}}_C = \hat{\mathbf{M}} \frac{\partial}{\partial t} \hat{\mathbf{v}}_C + \hat{\mathbf{v}}_C \times \hat{\mathbf{M}} \hat{\mathbf{v}}_C = \hat{\mathbf{M}} \frac{\partial}{\partial t} \hat{\mathbf{v}}_C + \boldsymbol{\omega} \times \hat{\mathbf{M}} \hat{\mathbf{v}}_C. \quad (37)$$

Note that Eq. (36) and Eq. (37) define the same motor referred to different points.

4. ENERGY FUNCTION AND THE MOMENT-PRODUCT OPERATION

Lagrange's formulation of rigid body dynamic equations requires the calculation of the energy function and its derivatives. Since we use dual number representation, derivatives of functions with respect to dual variables are needed. In this section, we calculate the energy function using dual vectors and then develop the rules by which derivatives of energy with respect to dual variables are taken - a subject which, to the best of our knowledge, has not been previously discussed in the literature.

4.1 Moment-Product Operation

Mutual operation between force and velocity motors or between momentum and velocity motors requires different rules from the regular dual number algebra multiplication of two dual vectors. The corresponding physically correct operation for screws was defined as early as 1900 by Ball, [34]. For motor algebra it was termed by Dimentberg [2] as the mutual moment of two motors and by Von Mises (see [8, 31]) as the scalar multiplication of screws. The same operation is denoted either as the inner-product of dual numbers (motors), [17], or Klein form, [9, 35], where the notation is:

$$[\hat{\mathbf{a}}|\hat{\mathbf{b}}] = [\mathbf{a} + \varepsilon \mathbf{a}^\circ | \mathbf{b} + \varepsilon \mathbf{b}^\circ] = \mathbf{a} \mathbf{b}^\circ + \mathbf{a}^\circ \mathbf{b}. \quad (38)$$

We refer to this operation as *moment-product* throughout the paper to underline both its physical meaning as a mutual moment of two skew lines and its mathematical sense as the moment part of a dual number.

This operation has a clear physical meaning. A moment-product of a force motor and a velocity motor is a power. One half of the moment-product of a dual momentum and a dual velocity motor of a body is the body's kinetic energy. Similarly, the potential energy of a body is obtained. Next, we derive the kinetic and potential energy of a rigid body while writing the vectors in their proper dual form and making use of the dual mass operator.

Observing that the moment-product of force screw and elementary displacement screw results in an elementary change in the kinetic energy of a moving body, one can draw a dual expression of kinetic energy for an elementary mass,

$$\Delta \mathbf{K} = \int_{r_1}^{r_2} [\hat{\mathbf{f}} | d\hat{\mathbf{r}}] = \int_{r_1}^{r_2} \left[d\hat{m} \frac{d\hat{\mathbf{v}}}{dt} | d\hat{\mathbf{v}} \cdot dt \right] = \int_{r_1}^{r_2} [d\hat{m} \hat{\mathbf{v}} | \hat{\mathbf{v}}], \quad (39)$$

and then obtain by integration the kinetic energy, \mathbf{K} , of the whole rigid body. This results in the moment-product of two dual motors of a different nature - the velocity and the momentum motor:

$$\mathbf{K} = \frac{1}{2} m \mathbf{v}_C^2 + \frac{1}{2} \boldsymbol{\omega}^T \mathbf{I} \boldsymbol{\omega} = \frac{1}{2} [\hat{\mathbf{v}}_C | \hat{\mathbf{h}}_C]. \quad (40)$$

In a similar way and using the dual mass property one can write the expression of the potential energy of a body in a gravitational field, displaced by \mathbf{x}_C :

$$P = \left[m \frac{d}{d\varepsilon} \varepsilon \mathbf{g} | \varepsilon \mathbf{x}_C \right] = m \mathbf{g} \mathbf{x}_C. \quad (41)$$

Since the moment-product of two motors does not depend on a reference point ([2, 8, 17]), one can rewrite Eq. (40) for both velocity and momentum motors being referred to some point, A , of the body, not necessarily the mass center:

$$\mathbf{K} = \frac{1}{2} [\hat{\mathbf{v}}_A | \hat{\mathbf{h}}_A] = \frac{1}{2} m \mathbf{v}_A^2 + m \mathbf{d}_{CA} \cdot (\boldsymbol{\omega} \times \mathbf{v}_A) + \frac{1}{2} \boldsymbol{\omega} \cdot (\mathbf{I}_A \boldsymbol{\omega}). \quad (42)$$

The second term of the latter expression can be interpreted as a moment of the body's inertia about an instantaneous screw axis. Indeed, if O is a point on the screw axis, and $\mathbf{v}_O = \mathbf{p} \boldsymbol{\omega}$, where \mathbf{p} is the pitch of the motor, simple algebraic manipulations result in:

$$m \mathbf{d}_{CA} \cdot (\boldsymbol{\omega} \times \mathbf{v}_A) = m \mathbf{d}_{CA} \cdot [\boldsymbol{\omega} \times (\mathbf{v}_O + \mathbf{D}_{AO} \boldsymbol{\omega})] = -m \boldsymbol{\omega} \cdot (\mathbf{D}_{CA} \mathbf{D}_{AO} \boldsymbol{\omega}). \quad (43)$$

Substituting Eq. (43) for Eq. (42) and expanding expression for \mathbf{I}_A , one obtains:

$$\begin{aligned} \mathbf{K} &= \frac{1}{2} m \mathbf{v}_A^2 - m \boldsymbol{\omega} \cdot [\mathbf{D}_{CA} \mathbf{D}_{AO} \boldsymbol{\omega}] + \frac{1}{2} \boldsymbol{\omega} \cdot \{ [\mathbf{I}_C - m \mathbf{D}_{CA}^2] \boldsymbol{\omega} \} = \\ &= \frac{1}{2} m \mathbf{v}_A^2 + \frac{1}{2} \boldsymbol{\omega} \cdot \{ [\mathbf{I}_C - m \mathbf{D}_{CA}^2 - 2m \mathbf{D}_{CA} \mathbf{D}_{AO}] \boldsymbol{\omega} \} = \frac{1}{2} m \mathbf{v}_A^2 + \frac{1}{2} \boldsymbol{\omega} \cdot \{ [\mathbf{I}_C - m \mathbf{D}_{CA} \mathbf{D}_{CO} - m \mathbf{D}_{CA} \mathbf{D}_{AO}] \boldsymbol{\omega} \}. \end{aligned} \quad (44)$$

This expression gives the kinetic energy of a moving body from its velocity and inertia parameters measured about the instantaneous screw axis.

4.2 Moment-Product Derivative

In order to use the energy function to derive Lagrange's dynamics equations, it is necessary to obtain derivative rules of a real function (e.g., energy) with respect to dual variables.

The result of a moment-product operation, Eq. (38), is a real scalar function. If at least one of the multiplied elements is a function of a dual variable $\hat{x} = x + \varepsilon x^\circ$, then the result, u , of this operation depends on this variable:

$$u = u(x, x^\circ). \quad (45)$$

Since this function depends on two independent variables, x and x° , its differential is:

$$du(x, x^\circ) = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial x^\circ} dx^\circ. \quad (46)$$

Let us suppose that this differential is a moment-product of the derivative of a real function with respect to the dual variable, $\hat{d}u/d\hat{x} = g(x, x^\circ) + \varepsilon q(x, x^\circ)$, and a differential of dual variable $d\hat{x}$:

$$du(x, x^\circ) = \left[\frac{\hat{d}u}{d\hat{x}} | d\hat{x} \right] = [g + \varepsilon q | dx + \varepsilon dx^\circ] = g dx^\circ + q dx. \quad (47)$$

Eqs. (46) and (47) define the same differential of the function u , hence:

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial x^\circ} dx^\circ = g dx^\circ + q dx, \quad (48)$$

or

$$\left(\frac{\partial u}{\partial x^\circ} - g \right) dx^\circ + \left(\frac{\partial u}{\partial x} - q \right) dx = 0. \quad (49)$$

For the function u to be analytic, Eq. (49) should be satisfied for an arbitrary ratio of dx/dx° , hence:

$$g = \frac{\partial u}{\partial x^\circ} \quad \text{and} \quad q = \frac{\partial u}{\partial x}, \quad (50)$$

which gives the derivative of a real function with respect to a dual variable:

$$\frac{\hat{d}u}{d\hat{x}} = \frac{\partial u}{\partial x^\circ} + \varepsilon \frac{\partial u}{\partial x}. \quad (51)$$

Comparing the last equation with the regular expression for derivative of a dual analytic function with respect to dual variable, Eq. (14), one can see that they are, actually, defined by the same relation.

Note that our derivation shows that *if multiplication of dual numbers is in the form of moment-product, then every real function of a dual variable is analytic.*

This result will subsequently be used to derive Lagrange's formulation of rigid body dynamics.

4.3 Moment-Product Derivative Rules

It would be worthwhile to verify whether the derivative of a moment-product operation satisfies the same rules as the regular derivative of real function with respect to real variable.

The derivative of the sum of two real functions with respect to dual variables is obtained by:

$$\frac{\hat{d}(u+v)}{d\hat{x}} = \frac{\partial(u+v)}{\partial x^\circ} + \varepsilon \frac{\partial(u+v)}{\partial x} = \frac{\hat{d}u}{d\hat{x}} + \frac{\hat{d}v}{d\hat{x}}. \quad (52)$$

One can easily check that the regular derivative rules also hold for derivatives of multiplication and division of real functions with respect to dual variables:

$$\begin{aligned} \frac{\hat{d}(u \cdot v)}{d\hat{x}} &= \frac{\partial(u \cdot v)}{\partial x^\circ} + \varepsilon \frac{\partial(u \cdot v)}{\partial x} = \frac{\partial u}{\partial x^\circ} v + u \frac{\partial v}{\partial x^\circ} + \varepsilon \left(\frac{\partial u}{\partial x} v + u \frac{\partial v}{\partial x} \right) \\ &= \frac{\hat{d}u}{d\hat{x}} v + u \frac{\hat{d}v}{d\hat{x}}, \end{aligned} \quad (53)$$

$$\begin{aligned} \frac{\hat{d}\left(\frac{u}{v}\right)}{d\hat{x}} &= \frac{\partial\left(\frac{u}{v}\right)}{\partial x^\circ} + \varepsilon \frac{\partial\left(\frac{u}{v}\right)}{\partial x} = \frac{\frac{\partial u}{\partial x^\circ} \cdot v - u \cdot \frac{\partial v}{\partial x^\circ}}{v^2} + \varepsilon \frac{\frac{\partial u}{\partial x} \cdot v - u \cdot \frac{\partial v}{\partial x}}{v^2} \\ &= \frac{\left(\frac{\partial u}{\partial x^\circ} + \varepsilon \frac{\partial u}{\partial x}\right) \cdot v - u \cdot \left(\frac{\partial v}{\partial x^\circ} + \varepsilon \frac{\partial v}{\partial x}\right)}{v^2} = \frac{\hat{d}u/d\hat{x} \cdot v - u \cdot \hat{d}v/d\hat{x}}{v^2}. \end{aligned} \quad (54)$$

A derivative of composite function with respect to a dual variable is given by:

$$\frac{\hat{d}u[v(\hat{x})]}{d\hat{x}} = \frac{\partial u[v(\hat{x})]}{\partial x^\circ} + \varepsilon \frac{\partial u[v(\hat{x})]}{\partial x} = \frac{du}{dv} \frac{\partial v}{\partial x^\circ} + \varepsilon \frac{du}{dv} \frac{\partial v}{\partial x} = \frac{du}{dv} \frac{\hat{d}v}{d\hat{x}}. \quad (55)$$

A more complicated case is obtained when a real function is a composite function of a dual function of a dual variable, namely,

$$u = u[\hat{v}(\hat{x})]. \quad (56)$$

In order to differentiate this function, one must assume the analytic character of the dual function $\hat{v}(\hat{x})$, namely, $\hat{v}(\hat{x})$ is differentiable. It is well known [2], that a dual analytic function of dual variable satisfies the conditions, Eqs. (12)-(15):

$$\hat{v}(\hat{x}) = v + \varepsilon v^\circ = v(x) + \varepsilon [x^\circ v'(x) + \tilde{v}(x)], \quad (57)$$

where $\tilde{v}(x)$ is an arbitrary function of x .

The moment-product derivative of a composite function u , Eq. (56), subjected to condition (57), is:

$$\begin{aligned} \frac{\hat{d}u[\hat{v}(\hat{x})]}{d\hat{x}} &= \frac{\partial u[\hat{v}(\hat{x})]}{\partial x^\circ} + \varepsilon \frac{\partial u[\hat{v}(\hat{x})]}{\partial x} = \\ &= \frac{\partial u}{\partial v} \frac{\partial v}{\partial x^\circ} + \frac{\partial u}{\partial v^\circ} \frac{\partial v^\circ}{\partial x^\circ} + \varepsilon \left(\frac{\partial u}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial v^\circ} \frac{\partial v^\circ}{\partial x} \right), \end{aligned} \quad (58)$$

and by condition (13), the first term is zero and the second term is equal to $\frac{\partial u}{\partial v^\circ} \frac{\partial v^\circ}{\partial x}$, then:

$$\begin{aligned} \frac{\hat{d}u[\hat{v}(\hat{x})]}{d\hat{x}} &= \frac{\partial u}{\partial v^\circ} \frac{\partial v^\circ}{\partial x} + \varepsilon \left(\frac{\partial u}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial v^\circ} \frac{\partial v^\circ}{\partial x} \right) = \\ &= \left(\frac{\partial u}{\partial v^\circ} + \varepsilon \frac{\partial u}{\partial v} \right) \left(\frac{\partial v}{\partial x} + \varepsilon \frac{\partial v^\circ}{\partial x} \right) = \frac{\hat{d}u}{d\hat{v}} \frac{d\hat{v}}{d\hat{x}}, \end{aligned} \quad (59)$$

Hence, the derivative of a composite function with respect to a dual variable has the same rule as the regular derivative of a composite function.

4.4 Differentiability

Since the first derivative of a real function by a dual variable is a dual function, it is reasonable to assume that a second derivative exists if the first one is an analytic dual function of Eq. (12), namely:

$$\frac{\hat{d}u}{d\hat{x}} = \frac{\partial u}{\partial x^\circ} + \varepsilon \frac{\partial u}{\partial x} = v(x) + \varepsilon [x^\circ v'(x) + \tilde{v}(x)]. \quad (60)$$

In this case, all existing derivatives of high order are dual functions and are calculated by Eq. (15).

From Eq. (60) it follows that in order to have a second derivative, the real function must have the following form:

$$u = x^\circ v(x) + \int \tilde{v}(x) dx + C, \quad (61)$$

where C is a constant of integration. The function of such a form has as many derivatives as exist for function $v(x)$ or one more than those existing for function $\tilde{v}(x)$.

It is interesting to examine the moment-product derivative of a real function u , from which the "moment-product integral" can be obtained. For example, the dual sine, cosine, and exponential functions can be considered as the moment-product derivative of the following real functions:

$$\begin{aligned} u = x^\circ \sin x &\Rightarrow \frac{\hat{d}u}{d\hat{x}} = \sin \hat{x} = \sin x + \varepsilon x^\circ \cos x, \\ u = x^\circ \cos x &\Rightarrow \frac{\hat{d}u}{d\hat{x}} = \cos \hat{x} = \cos x - \varepsilon x^\circ \sin x, \\ u = x^\circ e^x &\Rightarrow \frac{\hat{d}u}{d\hat{x}} = \exp(\hat{x}) = e^x + \varepsilon x^\circ e^x. \end{aligned} \quad (62)$$

Note that the same dual functions are obtained by taking dual derivatives of corresponding dual analytic functions:

$$\frac{\hat{d}(\cos \hat{x})}{d\hat{x}} = -\sin \hat{x}, \quad \frac{\hat{d}(\sin \hat{x})}{d\hat{x}} = \cos \hat{x}, \quad \frac{\hat{d}(\exp(\hat{x}))}{d\hat{x}} = \exp(\hat{x}). \quad (63)$$

One can draw the interesting conclusion that a dual derivative of a dual analytic function, Eq. (57), and a moment-product derivative of a dual part of the same analytic function coincide:

$$\begin{aligned} \frac{d}{d\hat{x}} \{v(x) + \varepsilon [x^\circ v'(x) + \tilde{v}(x)]\} &= \frac{\hat{d}}{d\hat{x}} [x^\circ v'(x) + \tilde{v}(x)] = \\ &v'(x) + \varepsilon [x^\circ v''(x) + \tilde{v}'(x)]. \end{aligned} \quad (64)$$

Finally, we compare the derivatives of the moment-product and the regular dual number algebra product of two dual analytic functions with respect to a dual variable.

The derivative of a moment-product of two analytic functions is defined using (53) and (57) as:

$$\begin{aligned} \frac{\hat{d}}{d\hat{x}}[u+\varepsilon u^\circ | v+\varepsilon v^\circ] &= \frac{\hat{d}}{d\hat{x}}[x^\circ(uv'+u'v)+(\tilde{u}v+u\tilde{v})]= \\ & uv'+u'v+\varepsilon\left[x^\circ\frac{d}{dx}(uv'+u'v)+\frac{d}{dx}(\tilde{u}v+u\tilde{v})\right]. \end{aligned} \quad (65)$$

The derivative of a regular dual number algebra product of two dual functions is:

$$\begin{aligned} \frac{d}{d\hat{x}}[(u+\varepsilon u^\circ)\cdot(v+\varepsilon v^\circ)] &= \\ [u'+\varepsilon(x^\circ u''+\tilde{u}')] \cdot (v+\varepsilon x^\circ v') + (u+\varepsilon u^\circ) \cdot [v'+\varepsilon(x^\circ v''+\tilde{v}')] &= \\ uv'+u'v+\varepsilon\left[x^\circ\frac{d}{dx}(uv'+u'v)+\frac{d}{dx}(\tilde{u}v+u\tilde{v})\right]. \end{aligned} \quad (66)$$

Hence, the derivatives of the moment and regular products of two dual analytic functions with respect to dual variables are equal to each other.

5. LAGRANGE'S DUAL DYNAMIC EQUATION

The above results are used next to derive the dual form of Lagrange's dynamic equation. Expressing the generalized coordinates in a dual form, $\hat{q}_i = \theta_i + \varepsilon d_i$, the energy is obtained as a moment-product of two dual functions. The dual generalized force, $\hat{Q}_i = f_i + \varepsilon \tau_i$, is obtained by differentiating the real energy function with respect to dual variable according to the moment-product derivative rule, Eq. (51):

$$\hat{Q}_i = \frac{d}{dt} \left(\frac{\hat{\partial}K}{\partial \hat{q}_i} \right) - \frac{\hat{\partial}K}{\partial \hat{q}_i} + \frac{\hat{\partial}P}{\partial \hat{q}_i} = \frac{d}{dt} \left(\frac{\partial K}{\partial d_i} \right) - \frac{\partial K}{\partial d_i} + \frac{\partial P}{\partial d_i} + \varepsilon \left[\frac{d}{dt} \left(\frac{\partial K}{\partial \hat{q}_i} \right) - \frac{\partial K}{\partial q_i} + \frac{\partial P}{\partial q_i} \right]. \quad (67)$$

If a generalized linear coordinate is properly represented as a pure dual number and a generalized rotational coordinate is properly represented as a pure real one, then, according to the moment-product derivative rule, the derivative of the energy with respect to the dual generalized coordinate yields a pure dual number in the case of moment, and a pure real number in the case of force. It should be mentioned that in Brodsky and Shoham's paper [7], the kinetic energy of a moving rigid body was derived by integration of a scalar product of dual force and infinitesimal motion, which resulted in a pure dual number. By taking derivatives of the energy with respect to real and dual variables, force and moment were correctly obtained in two distinct equations. In the present paper, the moment-product is used to yield the energy from which, by applying the moment-product derivative rules, the dual generalized force is obtained in only one equation.

5.1 Moment-Product Derivative of Kinetic Energy

As shown above, the energy of a rigid body is a moment-product of velocity and momentum motors. Since a momentum motor is itself a product of the dual inertia operator and the velocity motor, the resulting derivative is substantially simplified:

$$\frac{d}{d\hat{x}}[\hat{\mathbf{u}}|\hat{\mathbf{M}}\hat{\mathbf{v}}] = \left[\frac{d\hat{\mathbf{u}}}{d\hat{x}} \cdot \hat{\mathbf{M}}\hat{\mathbf{v}} \right] + \left[\frac{d\hat{\mathbf{v}}}{d\hat{x}} \cdot \hat{\mathbf{M}}\hat{\mathbf{u}} \right]. \quad (68)$$

In our case when vector-functions $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are the same:

$$\frac{d}{d\hat{x}}[\hat{\mathbf{v}}|\hat{\mathbf{M}}\hat{\mathbf{v}}] = 2\left[\frac{d\hat{\mathbf{v}}}{d\hat{x}}\cdot\hat{\mathbf{M}}\hat{\mathbf{v}}\right]. \quad (69)$$

In the case when vector $\hat{\mathbf{u}}$ is not a function of dual variable \hat{x} , Eq. (68) gives:

$$\frac{d}{d\hat{x}}[\hat{\mathbf{u}}|\hat{\mathbf{M}}\hat{\mathbf{v}}] = \left[\frac{d\hat{\mathbf{v}}}{d\hat{x}}\cdot\hat{\mathbf{M}}\hat{\mathbf{u}}\right], \quad \hat{\mathbf{u}} \neq \hat{\mathbf{u}}(\hat{x}). \quad (70)$$

In practice $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ may be measured in different coordinate systems. Then, assuming that the transformation matrix between these systems also depends on a dual variable \hat{x} , one obtains:

$$\frac{d}{d\hat{x}}[\hat{\mathbf{u}}|\hat{\mathbf{A}}\hat{\mathbf{M}}\hat{\mathbf{v}}] = \left[\frac{d}{d\hat{x}}(\hat{\mathbf{u}}\hat{\mathbf{A}})\cdot\hat{\mathbf{M}}\hat{\mathbf{v}}\right] + \left[\frac{d\hat{\mathbf{v}}}{d\hat{x}}\cdot\hat{\mathbf{M}}(\hat{\mathbf{u}}\hat{\mathbf{A}})\right]. \quad (71)$$

If the above expression is differentiated with respect to a real variable, then the result is:

$$\frac{d}{dx}[\hat{\mathbf{u}}|\hat{\mathbf{M}}\hat{\mathbf{v}}] = \left[\frac{d\hat{\mathbf{u}}}{dx}\cdot\hat{\mathbf{M}}\hat{\mathbf{v}}\right] + \left[\frac{d\hat{\mathbf{v}}}{dx}\cdot\hat{\mathbf{M}}\hat{\mathbf{u}}\right]. \quad (72)$$

These moment-product derivative rules are applied in the next section to analyze the dual dynamics in Lagrange's formulation.

6. DERIVATION OF NEWTON-EULER'S EQUATIONS FROM LAGRANGE'S EQUATIONS

The equivalence between the dual Newton-Euler and Lagrange formulations is shown next. It seems to us that it is worth discussing this obvious relation here to draw one's attention to the straightforward relation between the two formulations once dual numbers and the above-developed derivative rules, Eqs. (51) and (69), are used.

We start this derivation by calculating the energy of a rigid body, without loss of generality, about its mass center, Eq. (40):

$$\mathbf{K} = \frac{1}{2}[\hat{\mathbf{v}}_c|\hat{\mathbf{M}}\hat{\mathbf{v}}_c]. \quad (73)$$

Defining generalized coordinate as \hat{q}_i , namely:

$$\hat{q}_i = \theta_i + \varepsilon d_i, \quad (74)$$

then, the i -th Lagrange equation of motion corresponding to generalized coordinate \hat{q}_i has the following form:

$$\frac{d}{dt}\left(\frac{\partial\hat{K}}{\partial\dot{\hat{q}}_i}\right) - \frac{\partial\hat{K}}{\partial\hat{q}_i} + \frac{\partial\hat{P}}{\partial\hat{q}_i} = \hat{\tau}_i. \quad (75)$$

Let us start with rearranging the first term of (75) by using an expression for kinetic energy (73) and derivative rule (69), (analyticity of dual velocity as a function of both dual variable and time derivative of dual variable is assured). Simple substitutions lead to:

$$\frac{d}{dt}\left(\frac{\partial\hat{K}}{\partial\dot{\hat{q}}_i}\right) = \frac{d}{dt}\left(\frac{\partial\hat{\mathbf{v}}_c}{\partial\dot{\hat{q}}_i}\cdot\hat{\mathbf{M}}\hat{\mathbf{v}}_c\right) = \frac{d}{dt}\left(\frac{\partial\hat{\mathbf{v}}_c}{\partial\dot{\hat{q}}_i}\right)\cdot\hat{\mathbf{M}}\hat{\mathbf{v}}_c + \frac{\partial\hat{\mathbf{v}}_c}{\partial\dot{\hat{q}}_i}\cdot\frac{d\hat{\mathbf{M}}\hat{\mathbf{v}}_c}{dt}. \quad (76)$$

The second term of Eq. (75) can also be rewritten using Eq. (69):

$$\frac{\hat{\partial}K}{\partial\hat{q}_i} = \frac{\partial\hat{\mathbf{v}}_c}{\partial\hat{q}_i} \cdot \hat{\mathbf{M}}\hat{\mathbf{v}}_c. \quad (77)$$

Calculating now a sum of Eqs. (76) and (77), one obtains:

$$\frac{d}{dt} \left(\frac{\hat{\partial}K}{\partial\dot{\hat{q}}_i} \right) - \frac{\hat{\partial}K}{\partial\hat{q}_i} = \left[\frac{d}{dt} \left(\frac{\partial\hat{\mathbf{v}}_c}{\partial\dot{\hat{q}}_i} \right) - \frac{\partial\hat{\mathbf{v}}_c}{\partial\hat{q}_i} \right] \cdot \hat{\mathbf{M}}\hat{\mathbf{v}}_c + \frac{\partial\hat{\mathbf{v}}_c}{\partial\hat{q}_i} \cdot \frac{d\hat{\mathbf{M}}\hat{\mathbf{v}}_c}{dt}, \quad (78)$$

but the term in square brackets vanishes:

$$\frac{d}{dt} \left(\frac{\partial\hat{\mathbf{v}}_c}{\partial\dot{\hat{q}}_i} \right) - \frac{\partial\hat{\mathbf{v}}_c}{\partial\hat{q}_i} = 0. \quad (\text{Proof is given in Appendix}) \quad (79)$$

Physically it can be explained as follows: The velocity of a rigid body obtained through the given constraints is the only one kinetically possible; hence, it must satisfy the Lagrange equation (79).

Hence, Eq. (78) reduces to the form:

$$\frac{d}{dt} \left(\frac{\hat{\partial}K}{\partial\dot{\hat{q}}_i} \right) - \frac{\hat{\partial}K}{\partial\hat{q}_i} = \frac{\partial\hat{\mathbf{v}}_c}{\partial\hat{q}_i} \cdot \frac{d\hat{\mathbf{M}}\hat{\mathbf{v}}_c}{dt}. \quad (80)$$

The right hand side of Eq. (80) contains the Newton-Euler formulation in dual form expressed in a form of Eq. (37):

$$\frac{d\hat{\mathbf{M}}\hat{\mathbf{v}}_c}{dt} = \hat{\mathbf{F}}_c, \quad (81)$$

where $\hat{\mathbf{F}}_c$ is a dual force acting on the rigid body at the mass center.

The first term in the right hand side of Eq. (80), defined as

$$\frac{\partial\hat{\mathbf{v}}_c}{\partial\dot{\hat{q}}_i} = {}^c\hat{\mathbf{t}}_i, \quad (82)$$

is a unit dual vector along the i-th dual coordinate.

A scalar multiplication of dual force defined by Eq. (81) and unit dual vector, Eq. (82), gives the projection of a dual force $\hat{\mathbf{F}}_c$ on the direction of i-th general coordinate axis.

Taking also into account the potential energy component of a dual force, one finally has:

$$\frac{d}{dt} \left(\frac{\hat{\partial}K}{\partial\dot{\hat{q}}_i} \right) - \frac{\hat{\partial}K}{\partial\hat{q}_i} + \frac{\hat{\partial}P}{\partial\dot{\hat{q}}_i} = {}^c\hat{\mathbf{t}}_i \cdot \frac{d\hat{\mathbf{M}}\hat{\mathbf{v}}_c}{dt} + \frac{\hat{\partial}P}{\partial\dot{\hat{q}}_i} = \hat{f}_i. \quad (83)$$

Hence, a dual form of i-th Lagrange equation of a rigid body motion is a projection of the Newton-Euler equation on the direction of i-th general coordinate axis.

An important conclusion can be drawn considering numerical complexity aspects. Numerical algorithms which are based on Lagrange's approach and calculate each term of the Lagrange's equations of motion, actually calculate mutually canceling terms Eq. (79).

Two comments are in order:

In the case when i -th generalized coordinate presents only rotation or translation ($\hat{q}_i = \theta_i$ or $\hat{q}_i = d_i$), Eq. (80) has the following form (Eq. (79) is still satisfied):

$$\frac{d}{dt} \left(\frac{\partial K}{\partial \dot{q}_i} \right) - \frac{\partial K}{\partial q_i} = \left[\frac{\partial \hat{\mathbf{v}}_c}{\partial \dot{q}_i} \frac{d\hat{\mathbf{M}}\hat{\mathbf{v}}_c}{dt} \right], \quad (84)$$

which results in pure moment or force, respectively.

The second remark concerns a serial manipulator consisting of a number of links. Using Jacobian matrix form of a dual velocity expression, one can rewrite the right hand side of Eq. (83) in the form:

$$\hat{\boldsymbol{\tau}} = \sum_{i=1}^N \hat{\mathbf{J}}_i \cdot \frac{d\hat{\mathbf{M}}_i \hat{\mathbf{v}}_{c_i}}{dt} + \mathbf{p}, \quad (85)$$

where $\hat{\boldsymbol{\tau}} = [\hat{\tau}_1 \ \hat{\tau}_2 \ \dots \ \hat{\tau}_N]^T$ is a vector of dual generalized forces along the manipulator's joints,

$\hat{\mathbf{J}}_i$ is the dual Jacobian matrix of link i at its center of mass and $\mathbf{p} = \left[\frac{\partial P}{\partial \hat{q}_1} \ \frac{\partial P}{\partial \hat{q}_2} \ \dots \ \frac{\partial P}{\partial \hat{q}_N} \right]^T$

is the vector of dual potential forces in joints.

Eq. (85) is the basic term of an algorithm based on D'Alembert's approach and virtual work principle, [36], a dual formulation of which was obtained in [7]. This shows the equivalence between all three formulations of a manipulator dynamics algorithms, namely, Newton-Euler, Lagrange and D'Alembert formulations.

7. ILLUSTRATIVE EXAMPLE

As an example of dual numbers application to rigid body dynamics, consider the four degrees-of-freedom robot, shown in Fig. 2. Since this robot contains two cylindrical pairs, it can be perceived as having only two-dual-degrees-of-freedom.

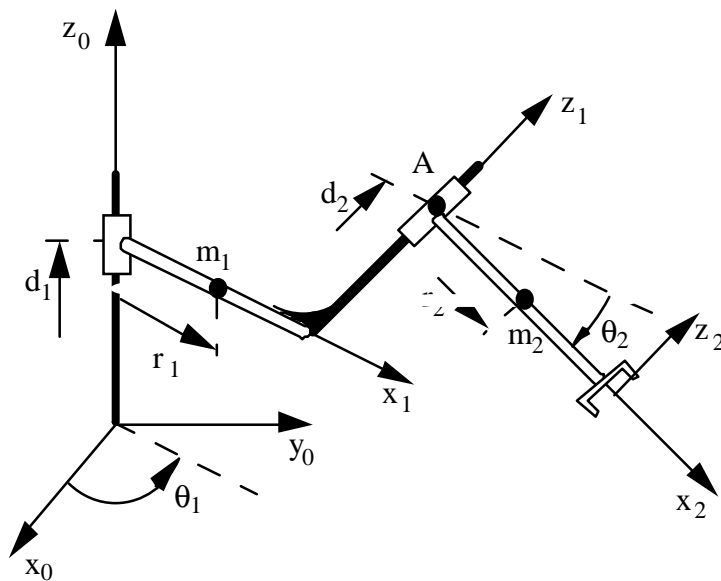


Fig. 2. Two dual (four real) degrees-of-freedom robot manipulator

The dual generalized coordinates of this robot are:

$$\hat{q}_1 = \theta_1 + \varepsilon d_1, \quad \hat{q}_2 = \theta_2 + \varepsilon d_2 \quad (86)$$

Using Denavite-Hartenberg notation to attach the coordinate systems, one obtains the following dual transformation matrices between the link attached coordinate systems (translated to respective center of mass) and joint attached ones:

$$\begin{aligned} \hat{\mathbf{A}}_0^1 &= \begin{bmatrix} \hat{c}_1 & \hat{s}_1 & 0 \\ -\varepsilon r_1 s_1 & \varepsilon r_1 c_1 & -1 \\ -\hat{s}_1 & \hat{c}_1 & \varepsilon r_1 \end{bmatrix}, & \hat{\mathbf{A}}_1^2 &= \begin{bmatrix} \hat{c}_2 & \hat{s}_2 & 0 \\ \varepsilon r_2 s_2 & -\varepsilon r_2 c_2 & 1 \\ \hat{s}_2 & -\hat{c}_2 & -\varepsilon r_2 \end{bmatrix}, \\ \hat{\mathbf{A}}_0^2 &= \begin{bmatrix} \hat{c}_1 \hat{c}_2 - \varepsilon \ell_1 s_1 s_2 & \hat{s}_1 \hat{c}_2 + \varepsilon \ell_1 c_1 s_2 & -\hat{s}_2 \\ -\hat{s}_1 + \varepsilon r_2 c_1 s_2 & \hat{c}_1 + \varepsilon r_2 s_1 s_2 & \varepsilon (\ell_1 + r_2 c_2) \\ \hat{c}_1 \hat{s}_2 + \varepsilon s_1 (\ell_1 c_2 + r_2) & \hat{s}_1 \hat{s}_2 - \varepsilon c_1 (\ell_1 c_2 + r_2) & \hat{c}_2 \end{bmatrix}. \end{aligned} \quad (87)$$

The velocity motors of the links' centers-of-mass are:

$$\begin{aligned} \hat{\mathbf{v}}_1 &= \begin{bmatrix} 0 \\ -1 \\ \varepsilon r_1 \end{bmatrix} \dot{\theta}_1 = \begin{bmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ -\dot{d}_1 \\ r_1 \dot{\theta}_1 \end{bmatrix}, \\ \hat{\mathbf{v}}_2 &= \begin{bmatrix} -\hat{s}_2 \\ \varepsilon (\ell_1 + r_2 c_2) \\ \hat{c}_2 \end{bmatrix} \dot{\theta}_1 + \begin{bmatrix} 0 \\ 1 \\ -\varepsilon r_2 \end{bmatrix} \dot{\theta}_2 = \begin{bmatrix} -s_2 \dot{\theta}_1 \\ \dot{\theta}_2 \\ c_2 \dot{\theta}_1 \end{bmatrix} + \varepsilon \begin{bmatrix} -s_2 \dot{d}_1 - d_2 c_2 \dot{\theta}_1 \\ r_2 c_2 \dot{\theta}_1 + \dot{d}_2 + \ell_1 \dot{\theta}_1 \\ c_2 \dot{d}_1 - d_2 s_2 \dot{\theta}_1 - r_2 \dot{\theta}_2 \end{bmatrix}. \end{aligned} \quad (88)$$

For simplification, we assume that links-attached coordinate system axes are principal, hence, the dual inertia operators become:

$$\hat{\mathbf{M}}_i = \begin{bmatrix} m_i \frac{d}{d\varepsilon} + \varepsilon I_{ix} & 0 & 0 \\ 0 & m_i \frac{d}{d\varepsilon} + \varepsilon I_{iy} & 0 \\ 0 & 0 & m_i \frac{d}{d\varepsilon} + \varepsilon I_{iz} \end{bmatrix}, \quad i=1,2. \quad (89)$$

The momentum motors of the first and the second link with respect to their centers of mass are calculated using Eq. (33) ($\hat{\mathbf{H}}_i = \hat{\mathbf{M}}_i \hat{\mathbf{v}}_i$):

$$\hat{\mathbf{H}}_1 = m_1 \begin{bmatrix} 0 \\ -\dot{d}_1 \\ r_1 \dot{\theta}_1 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ -I_{1y} \dot{\theta}_1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{H}}_2 = m_2 \begin{bmatrix} -s_2 \dot{d}_1 - d_2 c_2 \dot{\theta}_1 \\ r_2 c_2 \dot{\theta}_1 + \dot{d}_2 + \ell_1 \dot{\theta}_1 \\ c_2 \dot{d}_1 - d_2 s_2 \dot{\theta}_1 - r_2 \dot{\theta}_2 \end{bmatrix} + \varepsilon \begin{bmatrix} -I_{2x} s_2 \dot{\theta}_1 \\ I_{2y} \dot{\theta}_2 \\ I_{2z} c_2 \dot{\theta}_1 \end{bmatrix}. \quad (90)$$

If the momentum motor is calculated about an arbitrary point, Eq. (32) is used. For example, calculating the dual momentum motor of the second link about point A one obtains:

$$\begin{aligned} \hat{\mathbf{H}}_{2A} &= \hat{\mathbf{D}}_{A,2} \hat{\mathbf{H}}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon r_2 \\ 0 & \varepsilon r_2 & 1 \end{bmatrix} \hat{\mathbf{H}}_2 = \\ & m_2 \begin{bmatrix} -s_2 \dot{d}_1 - d_2 c_2 \dot{\theta}_1 \\ r_2 c_2 \dot{\theta}_1 + \dot{d}_2 + \ell_1 \dot{\theta}_1 \\ c_2 \dot{d}_1 - d_2 s_2 \dot{\theta}_1 - r_2 \dot{\theta}_2 \end{bmatrix} + \varepsilon \begin{bmatrix} -I_{2x} s_2 \dot{\theta}_1 \\ I_{2y} \dot{\theta}_2 - r_2 m_2 (c_2 \dot{d}_1 - d_2 s_2 \dot{\theta}_1 - r_2 \dot{\theta}_2) \\ I_{2z} c_2 \dot{\theta}_1 + r_2 m_2 (r_2 c_2 \dot{\theta}_1 + \dot{d}_2 + \ell_1 \dot{\theta}_1) \end{bmatrix}, \end{aligned} \quad (91)$$

where $\hat{\mathbf{D}}_{A,2}$ is the dual motor transformation rule matrix, Eq. (8), defined by vector $\mathbf{d}_{A,2} = [r_2 \ 0 \ 0]^T$.

Velocity motor at the point A of the second link is calculated in the same way:

$$\hat{\mathbf{v}}_{2A} = \hat{\mathbf{D}}_{A,2} \hat{\mathbf{v}}_2 = \begin{bmatrix} -s_2 \dot{\theta}_1 \\ \dot{\theta}_2 \\ c_2 \dot{\theta}_1 \end{bmatrix} + \varepsilon \begin{bmatrix} -s_2 \dot{d}_1 - d_2 c_2 \dot{\theta}_1 \\ \dot{d}_2 + \ell_1 \dot{\theta}_1 \\ c_2 \dot{d}_1 - d_2 s_2 \dot{\theta}_1 \end{bmatrix}. \quad (92)$$

In order to calculate kinetic energy, one should use the moment-product of velocity and momentum motors, evaluated at an arbitrary point:

$$\mathbf{K} = \frac{1}{2} \sum_{i=1}^2 [\hat{\mathbf{v}}_i \mid \hat{\mathbf{H}}_i] = \frac{1}{2} [\hat{\mathbf{v}}_1 \mid \hat{\mathbf{H}}_1] + \frac{1}{2} [\hat{\mathbf{v}}_2 \mid \hat{\mathbf{H}}_2] = \frac{1}{2} [\hat{\mathbf{v}}_1 \mid \hat{\mathbf{H}}_1] + \frac{1}{2} [\hat{\mathbf{v}}_{2A} \mid \hat{\mathbf{H}}_{2A}], \quad (93)$$

which yields:

$$\begin{aligned} \mathbf{K} = & \frac{1}{2} m_1 (\dot{d}_1^2 + r_1^2 \dot{\theta}_1^2) + \frac{1}{2} I_{1y} \dot{\theta}_1^2 + \frac{1}{2} m_2 \left\{ \dot{d}_1^2 + [d_2^2 + (\ell_1 + r_2 c_2)^2] \dot{\theta}_1^2 + \dot{d}_2^2 + r_2^2 \dot{\theta}_2^2 \right. \\ & \left. + 2(\ell_1 + r_2 c_2) \dot{\theta}_1 \dot{d}_2 - 2r_2 c_2 \dot{\theta}_2 \dot{d}_1 + 2d_2 r_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \right\} + \frac{1}{2} (I_{2x} s_2^2 + I_{2z} c_2^2) \dot{\theta}_1^2 + \frac{1}{2} I_{2y} \dot{\theta}_2^2. \end{aligned} \quad (94)$$

Applying now the moment-product derivative, Eq. (51), one obtains the dual Lagrange equations of motion:

$$\begin{aligned} \hat{\tau}_{1z} = & (m_1 + m_2) (\ddot{d}_1 + g) - m_2 r_2 c_2 \ddot{\theta}_2 + m_2 r_2 s_2 \ddot{\theta}_2^2 + \varepsilon \left\{ (I_{1y} + I_{2x} s_2^2 + I_{2z} c_2^2) \ddot{\theta}_1 + (I_{2x} - I_{2z}) s_2 c_2 \dot{\theta}_1 \dot{\theta}_2 + m_1 r_1^2 \ddot{\theta}_1 + \right. \\ & \left. m_2 [(d_2^2 + r_2^2 c_2^2 + \ell_1^2 + 2\ell_1 r_2 c_2) \ddot{\theta}_1 + (\ell_1 + r_2 c_2) \ddot{d}_2 + 2d_2 \dot{d}_2 \dot{\theta}_1 + r_2 d_2 c_2 \dot{\theta}_2^2 + r_2 d_2 s_2 \ddot{\theta}_2 - 2(\ell_1 r_2 + r_2^2 c_2) s_2 \dot{\theta}_1 \dot{\theta}_2] \right\}, \\ \hat{\tau}_{2z} = & m_2 (\ddot{d}_2 + (\ell_1 + r_2 c_2) \ddot{\theta}_1 - 2r_2 s_2 \dot{\theta}_1 \dot{\theta}_2 - \dot{\theta}_1^2 d_2) + \varepsilon \left\{ I_{2y} \ddot{\theta}_2 + (I_{2z} - I_{2x}) s_2 c_2 \dot{\theta}_1^2 + \right. \\ & \left. m_2 [r_2^2 \ddot{\theta}_2 - r_2 c_2 \ddot{d}_1 + 2r_2 s_2 \dot{d}_2 \dot{\theta}_1 + r_2 s_2 d_2 \ddot{\theta}_1 + (\ell_1 r_2 + r_2^2 c_2) s_2 \dot{\theta}_1^2 - g r_2 c_2] \right\}. \end{aligned} \quad (95)$$

The same result could be obtained by use of Eq. (85) presenting the Newton-Euler approach in combination with virtual work principle.

8. DISCUSSION AND CONCLUSIONS

Initially, his paper discusses the nature and the physical property of the dual inertia operator which, unlike the binor inertia matrix, allows connecting the dual velocity with the dual momentum spaces in a three-dimensional form. Note that the binor form of dynamics presumes transformation of binor and motors (velocity and momentum) in different spaces, 6x6 and 3x3 dual, respectively. With the dual inertia operator all transformations between different coordinate systems and different points are naturally embedded into dual motors space.

Using the dual inertia operator and the motor transformation rules, this paper develops then the general expression for dual momentum and uses it to obtain the Newton-Euler formulation of a rigid body dynamics in a three-dimensional dual form. Interestingly, the dual number representation seems to be a more complete one since it does not require a special term $m \mathbf{v}_A \times \mathbf{v}_C$ that is needed with the commonly real vector representation when moments are calculated about an arbitrary point (not about the center of mass or a stationary point).

With Lagrange's formulation, a special consideration is needed since dual generalized forces are obtained through derivatives of the energy function. Hence, derivative rules of a real function with respect to a dual variable are developed in this paper. In particular, derivative rules of a moment-product operation between the velocity and momentum motors, which is the physically correct operation yielding energy, are developed. It is shown that when analyticity is concerned, similar rules as in real variables for addition, multiplication and composite functions apply also in dual variables. Once these rules are developed, their application to the Lagrange equation yield the dual force (both force and moment) in only one equation. These derivative rules are used also to show how the dual Newton-Euler equations are derived from the dual Lagrange equation in a straightforward and simple way. The same is true for another multi-body dynamic algorithm of a serial manipulator which is based on D'Alembert and virtual work principles. All formulations calculate dual forces projected along the axes of the active joints. From the computational complexity point of view it is worth mentioning that Lagrange's equation includes mutually canceling terms.

One of the main objectives of this investigation is to examine the analogy between real and dual dynamics relations as appears in the derivations throughout the paper. This analogy cannot be obtained by standard application of the principle of transference for the dualization of a dynamic equation, namely, expanding from spherical to spatial dynamic.

Indeed, the principle of transference in kinematics is based on the dualization process [6], which expands a real function (scalar or vector) describing spherical kinematics into a dual analytic one describing spatial kinematics. For example, if a real function $u=u(\theta)$ is given, the corresponding dual analytic function will be:

$$\hat{u} = u(\theta) + \varepsilon d \frac{du(\theta)}{d\theta}.$$

The necessary condition allowing this transformation is that the real function must depend only on the real part of the corresponding dual variable. Apparently, dual momentum vector function cannot be obtained as a result of such a process. Its real part depends usually, on both real and dual parts of dual variables. However, using the Dual Inertia Operator we have come to the conclusion that there exists an analogy between real expressions of a rigid body dynamics describing spherical motion and dual ones describing spatial motion, as is shown in the following table. This analogy covers the calculation of dual momentum, kinetic energy and rigid body dynamics equations (both in Newton-Euler and Lagrange formulations).

For example, one can obtain the expression for dual momentum in the general case, Eq. (31) by substituting dual for real quantities in the expression for angular momentum (velocity motor for angular velocity vector, dual transformation matrices for real ones and dual inertia operator for the inertia matrix). This procedure is similar to the use of the principle of transference though it does not have the differential sense as does the kinematics dualizing routine.

RIGID BODY DYNAMICS EQUATIONS

Real Expressions	Dual Expressions
MOMENTUM OF A RIGID BODY	
Linear and Angular Momentum	Dual Momentum
<i>About Mass Center, Point C</i>	
<i>translational motion</i> $\mathbf{p} = m\mathbf{v}_C,$ <i>rotational motion</i> $\mathbf{H}_C = \mathbf{I}_C\boldsymbol{\omega}$	$\hat{\mathbf{H}}_C = \hat{\mathbf{M}}_C \hat{\mathbf{v}}_C$
<i>About an Arbitrary Point A</i>	
<i>translational motion</i> $\mathbf{p} = m\mathbf{v}_A + m\mathbf{r}_{CA} \times \boldsymbol{\omega},$ <i>rotational motion</i> $\mathbf{H}_A = (\mathbf{I}_C + \mathbf{D}_{AC}m\mathbf{D}_{CA})\boldsymbol{\omega}$	$\hat{\mathbf{H}}_A = \hat{\mathbf{D}}_{AC} \hat{\mathbf{M}}_C \hat{\mathbf{D}}_{CA} \hat{\mathbf{v}}_A$
<i>Velocity and Inertia Parameters Are Given in Different Coordinate Systems</i>	
<i>translational motion</i> $\mathbf{p} = m\mathbf{v}_C,$ <i>rotational motion</i> $\mathbf{H}_A = \mathbf{A}^T \mathbf{I}_A \mathbf{A} \boldsymbol{\omega}$	$\hat{\mathbf{H}}_A = \hat{\mathbf{A}}_{CA}^T \hat{\mathbf{M}}_C \hat{\mathbf{A}}_{CA} \hat{\mathbf{v}}_A$
KINETIC ENERGY	
<i>translational motion</i> $K = \frac{1}{2} \mathbf{v} \cdot m\mathbf{v}$ <i>rotational motion</i> $K = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I} \boldsymbol{\omega}$	$K = \frac{1}{2} [\hat{\mathbf{v}}_C \hat{\mathbf{M}}_C \hat{\mathbf{v}}] = \frac{1}{2} [\hat{\mathbf{v}}_C \hat{\mathbf{h}}_C]$
NEWTON-EULER DYNAMIC EQUATION	
<i>force</i> $\mathbf{f} = \frac{d\mathbf{p}}{dt},$ <i>moment</i> $\boldsymbol{\tau}_A = \frac{d\mathbf{H}_A}{dt} + m\mathbf{v}_A \times \mathbf{v}_C$	$\hat{\boldsymbol{\tau}}_A = \frac{d\hat{\mathbf{H}}_A}{dt}$
<i>force</i> $\mathbf{f} = \frac{d\mathbf{p}}{dt} = \frac{\partial m\mathbf{v}}{\partial t} + \boldsymbol{\omega} \times m\mathbf{v},$ <i>moment</i> $\boldsymbol{\tau}_A = \frac{\partial \mathbf{I}_A \boldsymbol{\omega}}{\partial t} + \boldsymbol{\omega} \times \mathbf{I}_A \boldsymbol{\omega} + m\mathbf{v}_A \times \mathbf{v}_C$	$\hat{\boldsymbol{\tau}}_A = \frac{d\hat{\mathbf{H}}_A}{dt} = \frac{\partial \hat{\mathbf{A}}(\hat{\mathbf{M}}_C \hat{\mathbf{v}})}{\partial t} + \hat{\mathbf{v}}_A \times \hat{\mathbf{A}}(\hat{\mathbf{M}}_C \hat{\mathbf{v}})$
LAGRANGE DYNAMIC EQUATION	
$Q_i = \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{q}_i} \right) - \frac{\partial K}{\partial q_i} + \frac{\partial P}{\partial q_i}$	$\hat{Q}_i = \frac{d}{dt} \left(\frac{\partial \hat{K}}{\partial \hat{q}_i} \right) - \frac{\partial \hat{K}}{\partial \hat{q}_i} + \frac{\partial \hat{P}}{\partial \hat{q}_i}$

REFERENCES

1. Clifford, W. K., *Proc. London Mathematic Society*, 1873, **4**, 381.
2. Dimentberg, F. M., *The Screw Calculus and its Applications in Mechanics*, (Izdat. 'Nauka', Moscow, USSR, 1965) English translation: AD680993, Clearinghouse for Federal and Scientific Technical Information.
3. Study, E., *Geometrie der Dynamen*, Leipzig, 1901.
4. Rooney, J., *Proc. 4th World Congress of IFTOMM5*, Newcastle-upon-Tyne, England, 1975, 1089.
5. Hsia, L. M., and Yang, A. T., *ASME J. of Mechanical Design*, 1981, **103**, 652.
6. Martinez, J. M. R., and Duffy, J., *Mechanism and Machine Theory*, 1993, **26**(1), 165.
7. Brodsky, V., and Shoham, M., *ASME J. of Mechanical Design*, 1994, **116**, 1089.
8. Brand, L., *Vector and Tensor Analysis*, 7th printing, John Wiley & Sons Inc., New York, 1958.
9. Chevallier, D. P., *Mechanism and Machine Theory*, 1996, **31**(1), 57.
10. Woo, L. S., and Freudenstein, F., *J. of Mechanisms*, 1970, **5**, 417.
11. Veldkamp, G. R., *Mechanism and Machine Theory*, 1976, **11**, 141.
12. Keler, M. L., *Environment and Planning B*, 1979, **6**, 403.
13. Pennock, G. R., and Yang, A. T., *ASME J. of Mechanisms, Transmissions, and Automation in Design*, 1983, **105**, 28-34.
14. McCarthy, J. M., *The Int. J. of Robotics Research*, 1986, **5**(2), 45.
15. Gu, Y. L., and Luh, J. Y. S., *IEEE J. of Robotics and Automation*, 1987, **RA-3**(6), 615.
16. Pradeep, A. K., Yoder, P. J., and Mukundan, R., *The Int. J. of Robotics Research*, 1989, **8**(5), 55.
17. Chevallier, D. P., *Mechanism and Machine Theory*, 1991, **26**(6), 613.
18. Gonzalez-Palacios, M. A., and Angeles, J., *Cam Synthesis*, Kluwer Academic Publishers, 1993.
19. Cheng, H. H., *Engineering with Computers*, 1994, **10**(4), 212.
20. Yang, A. T., *ASME J. of Engineering for Industry*, Series B, 1971, **93**, 27.
21. Yang, A. T., *Basic Questions of Design Theory*, ed. by W. R. Spillers, North-Holland Publishing Company, Amsterdam, 1974, 265.
22. Bagci, C., *ASME J. of Engineering for Industry*, 1972, **94**, 738.
23. Lee, I. P. J., and Soni, A. H., *J. of Mechanical Design*, 1979, **101**, 569.
24. Luh, J. Y. S., and Gu, Y. L., *Proc. 1984 Conf. Information Sciences and Systems*, Princeton, NJ, March 1984, 680.
25. Akselrod, B. V., *Mekhanika Tverdogo Tela (Mechanics of Solids)*, Allerton Press Inc., 1985, **20**(2), 79.
26. Yanzhu, L., *Acta Mechanica Sinica*, Science Press, Beijing, China, Allerton Press Inc., 1988, **4**(2).
27. Selig, J. M., *Prikladnaya Matematika i Mekhanika (J. of Applied Math. and Mech.)*, 1991, **55**(2).
28. Pennock, G. R., and Oncu, B. A., *ASME J. of Dyn. Sys. Meas. and Contr.*, 1992, **114**(2), 262.
29. Agrawal, S. K., *ASME J. of Mechanical Design*, 1993, **115**, 833.
30. Baklouti, M., and Castelain, J. M., *J. of Robotic Systems*, 1993, **10**(2), 271.

31. Wohlhart, K., *Computational Kinematics*, (Eds. J.-P. Merlet and B. Ravani), Kluwer Academic Publishers, 1995, 93.
32. Brodsky, V., and Shoham, M., *2nd ECPD Int. Conf. on Advanced Robotics, Intelligent Automation and Active Systems*, Vienna, Austria, 1996.
33. Dimentberg, F. M., *Theory of Hinted Spatial Mechanisms*, "Nauka", Moscow, USSR, 1982.
34. Ball, R. S., *A Treatise on the Theory of Screws*, Cambridge University Press, 1900.
35. Karger, A., Novak, J., *Space Kinematics and Lie Groups*, Gordon and Breach Science Publishers, 1985.
36. Shoham, M. and Srivatsan, R., *Robotics & Computer-Integrated Manufacturing*, 1992, **9**(3), 219.

APPENDIX

This Appendix proves that Lagrange's formulation includes mutually canceling terms, Eq. (79)

$$\frac{d}{dt} \left(\frac{\partial \hat{v}_c}{\partial \dot{\hat{q}}_i} \right) - \frac{\partial \hat{v}_c}{\partial \hat{q}_i} = 0 \quad (a1)$$

The n-th link dual velocity is (for a free rigid body virtual links can be considered):

$$\hat{v}_n = \sum_{j=1}^n {}^n \hat{A}_j \dot{\hat{\theta}}_j. \quad (a2)$$

The derivative of a dual rotational transformation matrix is:

$$\frac{\partial {}^i \hat{A}_{i-1}}{\partial \dot{\hat{\theta}}_i} = -(\hat{e}_i \times) {}^i \hat{A}_{i-1}, \quad (a3)$$

where ${}^i \hat{A}_{i-1}$ is a matrix transforming motors from i-1-th coordinate system to i-th ; $(\hat{e}_i \times)$ is a skew-symmetric cross-product matrix of dual unit vector \hat{e}_i of i-th rotation axis.

The derivative of velocity motor with respect to a dual variable is obtained by using Eqs. (a2) and (a3):

$$\frac{\partial \hat{v}_n}{\partial \dot{\hat{\theta}}_i} = \sum_{j=1}^n \frac{\partial {}^n \hat{A}_j}{\partial \dot{\hat{\theta}}_i} \hat{e}_j \dot{\hat{\theta}}_j = -\sum_{j=1}^{i-1} {}^n \hat{A}_i (\hat{e}_i \times) {}^i \hat{A}_j \hat{e}_j \dot{\hat{\theta}}_j. \quad (a4)$$

A derivative of velocity motor with respect to time-derivative of a dual variable is (see Eq. (a2)):

$$\frac{\partial \hat{v}_n}{\partial \dot{\hat{\theta}}_i} = {}^n \hat{A}_i \hat{e}_i. \quad (a5)$$

This derivative is a motor, since it is a dual transformation of a dual unit vector, and its time-derivative has to be calculated using Eq. (34):

$$\frac{d}{dt} \left(\frac{\partial \hat{v}_n}{\partial \dot{\hat{\theta}}_i} \right) = {}^n \hat{A}_i \dot{\hat{e}}_i + \hat{v}_n \times ({}^n \hat{A}_i \hat{e}_i). \quad (a6)$$

Now, the first and the second terms are calculated separately:

$${}^n \hat{\mathbf{A}}_i \dot{\hat{\mathbf{e}}}_i = \sum_{j=i}^n \frac{\partial {}^n \hat{\mathbf{A}}_j}{\partial \hat{\boldsymbol{\theta}}_j} \dot{\hat{\boldsymbol{\theta}}}_j \hat{\mathbf{e}}_i = - \sum_{j=i}^n {}^n \hat{\mathbf{A}}_j (\hat{\mathbf{e}}_j \times) {}^j \hat{\mathbf{A}}_i \hat{\mathbf{e}}_i \dot{\hat{\boldsymbol{\theta}}}_j, \quad (\text{a7})$$

and

$$\hat{\mathbf{v}}_n \times ({}^n \hat{\mathbf{A}}_i \dot{\hat{\mathbf{e}}}_i) = \left(\sum_{j=1}^n {}^n \hat{\mathbf{A}}_j \hat{\mathbf{e}}_j \dot{\hat{\boldsymbol{\theta}}}_j \right) \times ({}^n \hat{\mathbf{A}}_i \dot{\hat{\mathbf{e}}}_i) = \sum_{j=1}^{i-1} {}^n \hat{\mathbf{A}}_i ({}^i \hat{\mathbf{A}}_j \hat{\mathbf{e}}_j) \times \hat{\mathbf{e}}_i \dot{\hat{\boldsymbol{\theta}}}_j + \sum_{j=i}^n {}^n \hat{\mathbf{A}}_j (\hat{\mathbf{e}}_j \times) {}^j \hat{\mathbf{A}}_i \hat{\mathbf{e}}_i \dot{\hat{\boldsymbol{\theta}}}_j. \quad (\text{a8})$$

Substitution of Eqs. (a4), (a7) and (a8) into Eq. (a1) results in

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \hat{\mathbf{v}}_c}{\partial \dot{\hat{\mathbf{q}}}_i} \right) - \frac{\partial \hat{\mathbf{v}}_c}{\partial \hat{\mathbf{q}}_i} = & - \sum_{j=i}^n {}^n \hat{\mathbf{A}}_j (\hat{\mathbf{e}}_j \times) {}^j \hat{\mathbf{A}}_i \hat{\mathbf{e}}_i \dot{\hat{\boldsymbol{\theta}}}_j + \\ & \sum_{j=1}^{i-1} {}^n \hat{\mathbf{A}}_i ({}^i \hat{\mathbf{A}}_j \hat{\mathbf{e}}_j) \times \hat{\mathbf{e}}_i \dot{\hat{\boldsymbol{\theta}}}_j + \sum_{j=i}^n {}^n \hat{\mathbf{A}}_j (\hat{\mathbf{e}}_j \times) {}^j \hat{\mathbf{A}}_i \hat{\mathbf{e}}_i \dot{\hat{\boldsymbol{\theta}}}_j + \sum_{j=1}^{i-1} {}^n \hat{\mathbf{A}}_i (\hat{\mathbf{e}}_i \times) {}^i \hat{\mathbf{A}}_j \hat{\mathbf{e}}_j \dot{\hat{\boldsymbol{\theta}}}_j. \end{aligned} \quad (\text{a9})$$

The first term on the right side of (a9) is equal to the third one with an opposite sign; hence, their sum vanishes. The sum of the remaining two terms can be calculated as follows:

$$\begin{aligned} & \sum_{j=1}^{i-1} {}^n \hat{\mathbf{A}}_i ({}^i \hat{\mathbf{A}}_j \hat{\mathbf{e}}_j) \times \hat{\mathbf{e}}_i \dot{\hat{\boldsymbol{\theta}}}_j + \sum_{j=1}^{i-1} {}^n \hat{\mathbf{A}}_i (\hat{\mathbf{e}}_i \times) {}^i \hat{\mathbf{A}}_j \hat{\mathbf{e}}_j \dot{\hat{\boldsymbol{\theta}}}_j = \\ & \sum_{j=1}^{i-1} ({}^n \hat{\mathbf{A}}_j \hat{\mathbf{e}}_j) \times ({}^n \hat{\mathbf{A}}_i \hat{\mathbf{e}}_i) \dot{\hat{\boldsymbol{\theta}}}_j + \sum_{j=1}^{i-1} ({}^n \hat{\mathbf{A}}_i \hat{\mathbf{e}}_i) \times ({}^n \hat{\mathbf{A}}_j \hat{\mathbf{e}}_j) \dot{\hat{\boldsymbol{\theta}}}_j = 0. \end{aligned} \quad (\text{a10})$$

Hence, all the right hand side in Eq. (a9) vanishes, which proves our assumption.